Q1

(a) $T(s) = \frac{K}{s(s+10)} = K\frac{N(s)}{D(s)}$

* starting points $p_1 = 0$, $p_2 = -10$

* ending points asymptotes (∞, ∞)

* $\deg(N(s)) = 0 = m$
$\deg(D(s)) = 2 = n$

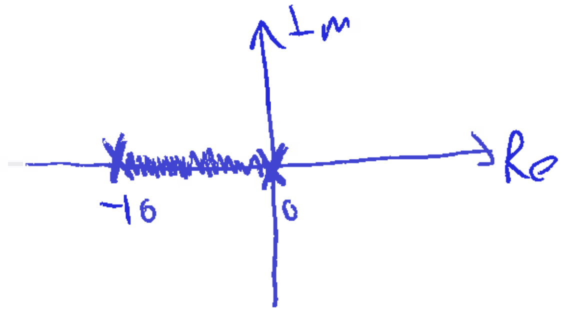
* num-branches $= \max(m,n) = \max(2,0) = 2$

* num-asymptotes $= |n-m| = |2-0| = 2$

* angle of asymptotes $= 90°, -90°$

* centroid $= \dfrac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{|n-m|} = \dfrac{[0+(-10)]}{2} = -5$

* locus on real axis



* Break in breakaway points

$$\left[\frac{d}{ds}(N(s))\right]D(s) - \left[\frac{d}{ds}(D(s))\right]N(s) = 0$$

$$0 \times s \times (s+10) - [2s+10]\cdot 1 = 0$$

$$-[2s+10] = 0 \Rightarrow 2s+10 = 0 \quad s = \frac{-5}{1} = -5$$

(breakaway point)

* Imaginary axis crossing

$$\frac{K}{s(s+10)} + 1 = 0 \quad \text{put } s = j\omega$$

$$\frac{K}{j\omega(j\omega+10)} + 1 = 0 \Rightarrow K + [-\omega^2 + j\omega 10] = 0$$

$$[K - \omega^2] + j\omega 10 = 0$$

(only solution $\omega = 0$ $K = 0$ (starting point) $\omega = 0 \Rightarrow s = j\omega = 0$)

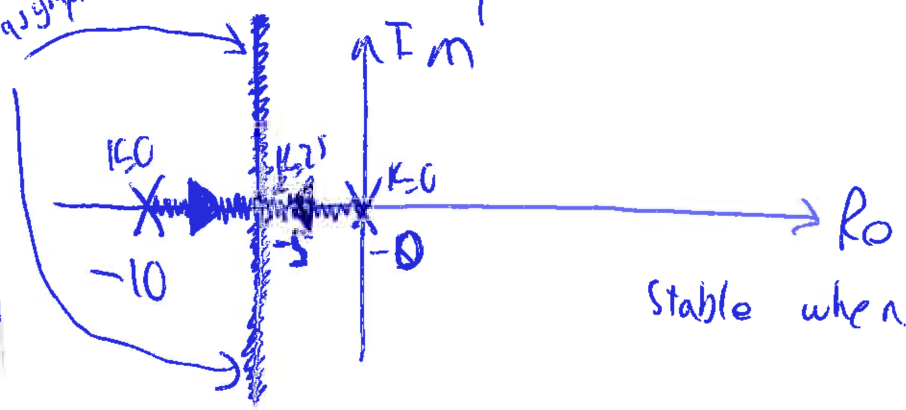
at break-away point $s = -5$

$$\frac{K}{s(s+10)} + 1 = 0$$

$$\frac{K}{-5[-5+10]} + 1 = 0$$

$$\frac{K}{(-5)(5)} + 1 = 0 \quad K = 25$$

Root-locus plot



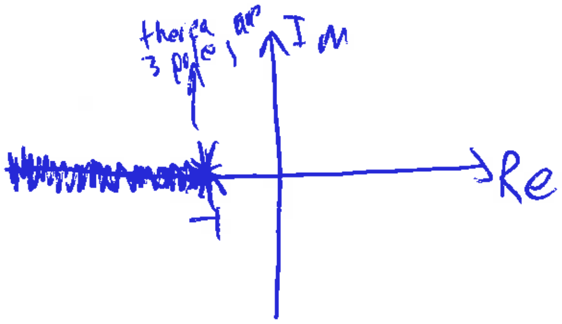
Stable when $0 < K < \infty$

(b) $T(s) = \dfrac{K}{(s+1)^3} = K\dfrac{N(s)}{D(s)}$ * starting point $p_1=-1$ $p_2=-1$ $p_3=-1$ * ending points asymptotes (∞,∞,∞)

* $\deg N(s) = 0 = m$
  $\deg D(s) = 3 = n$

* num_branches $= \max(m,n) = \max(3,0) = 3$
* num_asymptotes $= |n-m| = |3-0| = 3$
* angle of asymptotes $= 60^\circ, 180^\circ, 300^\circ$ [or -60°]
* centroid $= \dfrac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{|n-m|} = \dfrac{(-1)+(-1)+(-1)}{3} = -1$

* locus on real axis



* Break-in break-away points

$$\left[\frac{d}{ds}N(s)\right]D(s) - \left[\frac{d}{ds}D(s)\right]N(s) = 0$$

$$0 \times (s+1)^3 - 3\times(s+1)^2 \times 1 \times 1 = 0$$

$$-3(s+1)^2 = 0 \Rightarrow (s+1)^2 = 0 \quad s_1 = s_2 = -1$$

$$s=-1 \Rightarrow \frac{K}{(s+1)^3} + 1 = 0 \Rightarrow K=0 \text{ (starting point also)}$$

* starting and break-in break-away point is the same

* Imaginary axis crossing

$$\frac{K}{(s+1)^3} + 1 = 0 \quad (\text{put } s = j\omega)$$

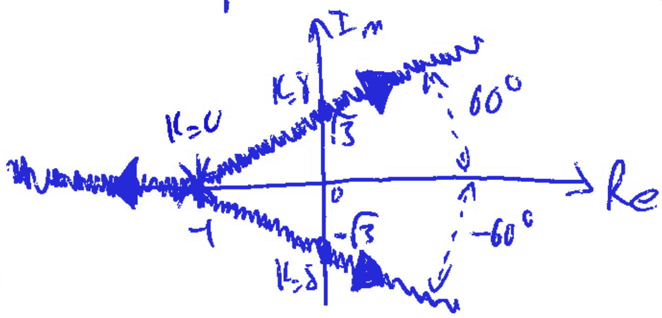
$$\frac{K}{(j\omega+1)^3} + 1 = 0 \quad K + [-j\omega^3 - 3\omega^2 + 3j\omega + 1] = 0 \Rightarrow [1+K-3\omega^2] + j\omega[3-\omega^2] = 0$$

$\omega = 0 \Rightarrow K = -1$ (not in root locus)

$\boxed{\omega = \mp\sqrt{3} \to 1+K-9=0 \Rightarrow K=8 \text{ (in root locus)}}$

↳ if $K=8$ $\quad \dfrac{K}{(s+1)^3} + 1 = 0 \Rightarrow \dfrac{8}{(s+1)^3} + 1 = 0 \Rightarrow 8 + s^3 + 3s^2 + 3s + 1 = 0$
↳ $s^3 + 3s^2 + 3s + 9 = 0$

Hence if $K=8$ root location $(s_1 = +\sqrt{3}j,\ s_2 = -\sqrt{3}j,\ s_3 = -3)$ $\quad (s^2+3)(s+3) = 0$



stable when $0 < K < 8$

(c) $T(s) = \dfrac{Ks}{(s-1)(s-2)} = K\dfrac{N(s)}{D(s)}$

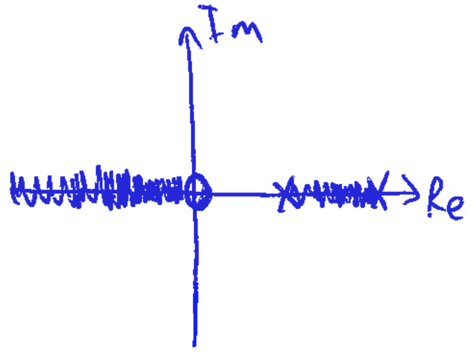
* starting points $p_1 = 1,\ p_2 = 2$
* ending points $z_1 = 0$, asymptote (∞)

$\deg(N(s)) = 1 = m$
$\deg(D(s)) = 2 = n$

* num_branches = $\max(m, n) = \max(2,1) = 2$
* num_asymptotes = $|n-m| = |2-1| = 0$
* angle of asymptotes = 180°
* centroid = $\dfrac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{|n-m|} = \dfrac{(1)+(2) - (0)}{|2-1|} = \dfrac{3}{1} = 3$

Page 3

locus on real axis



Break-in break-away points

$$\left[\frac{d}{ds}N(s)\right]D(s) - \left[\frac{d}{ds}D(s)\right]N(s) = 0$$

$$1\,(s-1)(s-2) - [2s-3]\,s = 0$$

$$s^2 - 3s + 2 - 2s^2 + 3s = 0$$

$$-s^2 + 2 = 0 \Rightarrow s^2 = 2$$

$s = \mp\sqrt{2}$ (possible break-in / break-away points)

At $s = \sqrt{2} \Rightarrow \dfrac{K\sqrt{2}}{(\sqrt{2}-1)(\sqrt{2}-2)} + 1 = 0$

$$K = \frac{(-1)(\sqrt{2}-1)\sqrt{2}[1-\sqrt{2}]}{\sqrt{2}} = (\sqrt{2}-1)^2 = 2+1-2\sqrt{2} = 3-2\sqrt{2} > 0 \text{ (in root locus)}$$

$s = -\sqrt{2} \Rightarrow \dfrac{K[-\sqrt{2}]}{[-\sqrt{2}-1][-\sqrt{2}-2]} + 1 = 0 \Rightarrow K = \dfrac{-1[\sqrt{2}+1][\sqrt{2}+2]}{[-\sqrt{2}]} = [\sqrt{2}+1]^2$

$K = 3+2\sqrt{2} > 0$ (in root locus)

Imaginary axis crossing

put $s = j\omega$

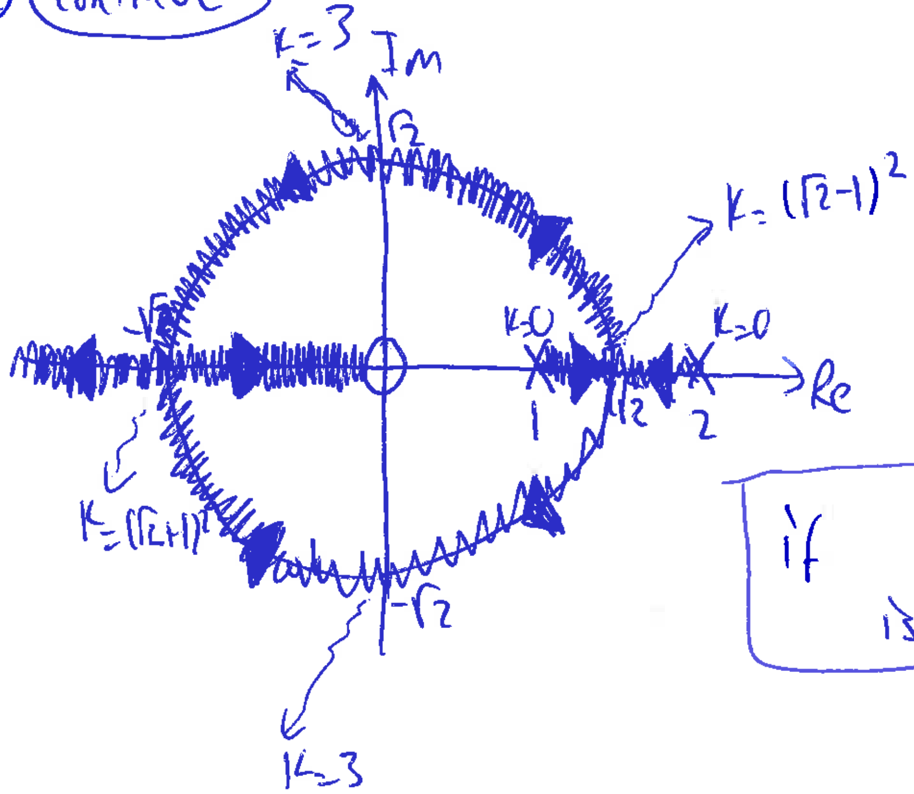
$$\frac{Ks}{(s-1)(s-2)} + 1 = 0 \Rightarrow \begin{cases} Ks + (s-1)(s-2) = 0 \\ s^2 + (K-3)s + 2 = 0 \end{cases}$$

put $s = j\omega$: $-\omega^2 + (K-3)j\omega + 2 = 0$

$(2-\omega^2) + j\omega(K-3)$

only solution $K = 3$ and $\omega = \mp j\sqrt{2}$ (in root locus)

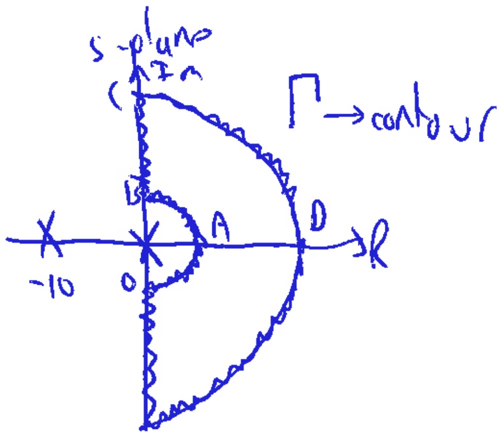
Q1 c continue



if $K > 3$ system is stable

(Q2) (a) $T(s) = \dfrac{K}{s(s+10)}$ $\quad N = z - p$

$p = 0$ [num of open loop poles of $T(s)$ in ORHP]



$$\Gamma = \begin{cases} AB, & s = re^{j\theta}, \ \theta: 0 \to 90, \ r \approx 0 \\ BC, & s = j\omega, \ \omega = 0 \to \infty \\ CD, & s = Re^{j\theta}, \ \theta = 90 \to 0, \ R \approx \infty \end{cases}$$

at AB $\quad s = re^{j\theta} \quad \theta; 0 \to 90$

$$T(s) = \frac{K}{re^{j\theta}(re^{j\theta}+10)} \approx \frac{K}{re^{j\theta} \times 10} \approx \underbrace{R}_{\infty}\, e^{-j\theta} \quad \angle T(s) = -\theta$$

∞ (a very big circle

if $\theta = 0 \Rightarrow \angle T(s) = 0°$

if $\theta = 45 \Rightarrow \angle T(s) = -45°$

if $\theta = 90 \Rightarrow \angle T(s) = -90°$ //

at BC $\quad s = j\omega$

$$T(s) = \frac{K}{j\omega(j\omega+10)} \cdot (10-j\omega) = \frac{K(10-j\omega)}{j\omega[10^2+\omega^2]} \cdot (j) = \frac{Kj[10-j\omega]}{(-1)\omega[10^2+\omega^2]} = \frac{-10Kj - K\omega}{\omega[10^2+\omega^2]} \ //$$

$$T(s) = j\underbrace{\frac{-10K}{\omega[10^2+\omega^2]}}_{Im} - \underbrace{\frac{K\omega}{\omega[10^2+\omega^2]}}_{Re}$$

	$\omega = 0$	$0 < \omega < \infty$	$\omega \to \infty$
$Re(T(s))$	$-\frac{K}{10}$	—	0
$Im(T(s))$	$-\infty$	—	0

at CD $\quad s = Re^{j\theta} \quad \theta: 90 \to 0$

$$T(s) \simeq \frac{K}{Re^{j\theta}(Re^{j\theta}+10)} \simeq \frac{K}{R^2 e^{2j\theta}} \simeq \underbrace{r}_{\simeq 0} e^{-j2\theta}$$

$$\angle T(s) = -2\theta$$

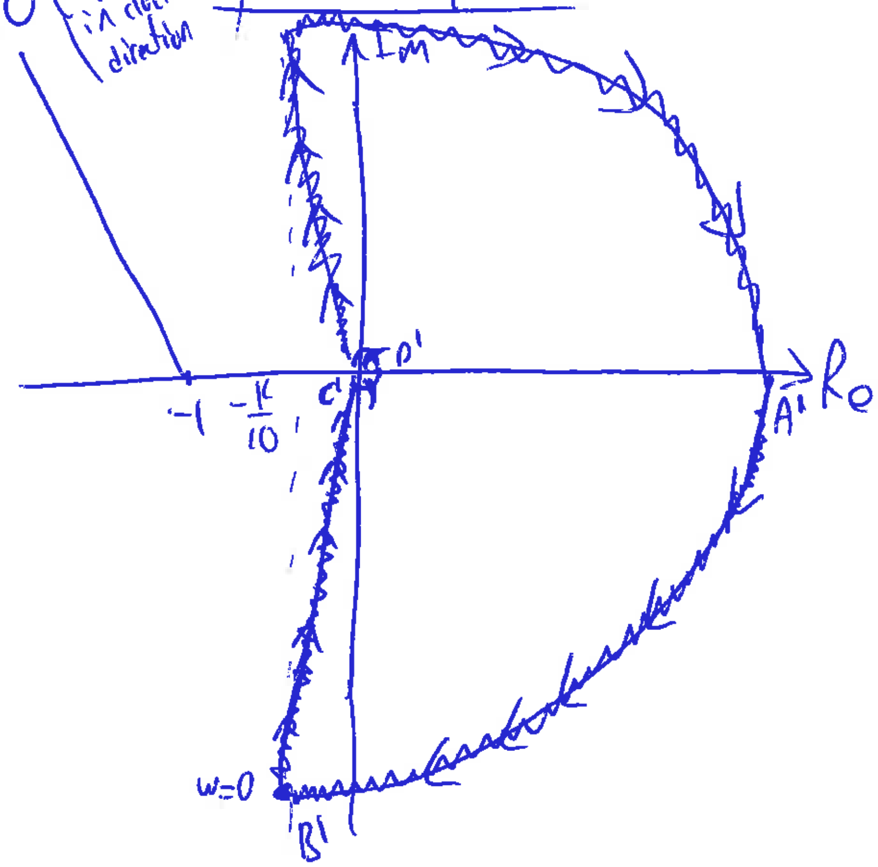
$\theta = 90 \rightarrow -2\theta = -180^0$

$\theta = 45 \Rightarrow -2\theta = -90^\circ$

$\theta = 0 \Rightarrow -2\theta = 0^\circ$ //

N=0 (encirclement of -1 in clockwise direction)

Nyquist plot



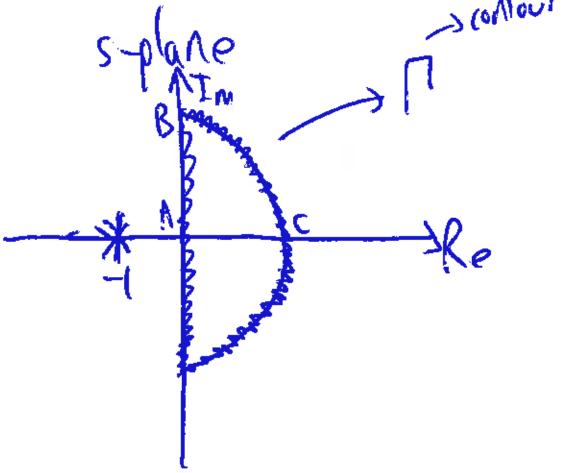
when $K > 0$ this system is stable

$N = Z - P$

$0 = Z - 0$ $\quad \boxed{Z = 0}$ (closed loop poles of $1+T(s)$ in ORHP)

(b) $T(s) = \dfrac{K}{(s+1)^3}$

$$\Gamma = \begin{cases} AB, & s = j\omega, \ \omega: 0 \to \infty \\ BC, & s = Re^{j\theta}, \ \theta: 90 \to 0, \ R \approx 0 \end{cases}$$



Page 7

at AB $s = j\omega$

$$T(s) = \frac{K}{(j\omega+1)^3} \cdot \frac{(1-j\omega)^3}{(1-j\omega)^3} = \frac{K[1-j\omega]^3}{[1+\omega^2]^3}$$

$P = 0$ (open loop poles of $T(s)$ in ORHP)

$$T(s) = \frac{K[1 - 3j\omega - 3\omega^2 + j\omega^3]}{(1+\omega^2)^3}$$

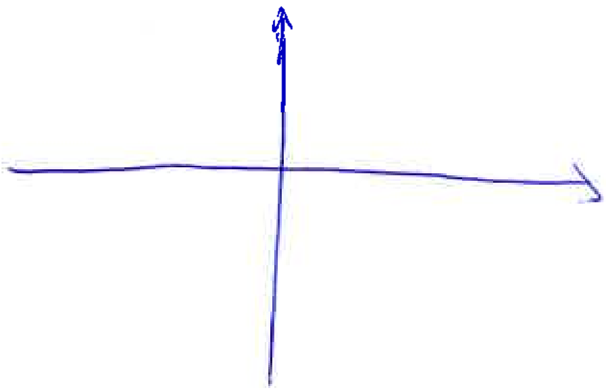
$$T(s) = \frac{K[1-3\omega^2]}{[1+\omega^2]^3} + j\frac{\omega[\omega^2-3]}{[1+\omega^2]^3}$$

	$\omega=0$	$0<\omega<\frac{1}{\sqrt{3}}$	$\omega=\frac{1}{\sqrt{3}}$	$\frac{1}{\sqrt{3}}<\omega<\sqrt{3}$	$\omega=\sqrt{3}$	[illegible]	$\omega\to\infty$
$Re(T(s))$	K	$+$	0	$-$	$-\frac{K}{8}$	$-$	0
$Im(T(s))$	0	$-$	$-\frac{\sqrt{3}}{8}K$	$-$	0	$+$	0

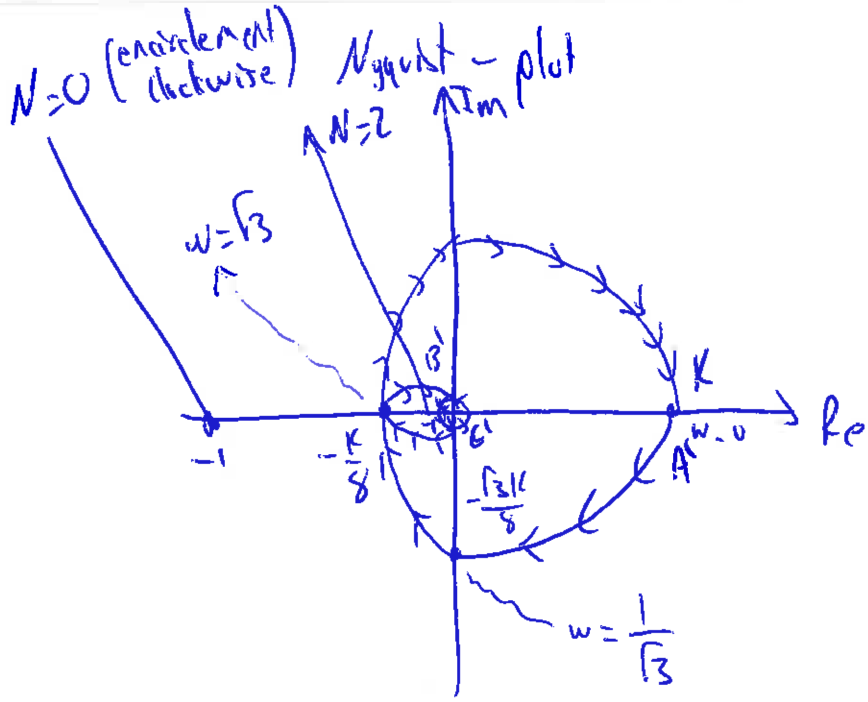
at BC $s = Re^{j\theta}$, $\theta: 90 \to 0$

$$T(s) = \frac{K}{(Re^{j\theta}+1)^3} \approx \frac{K}{R^3 e^{j3\theta}} \approx r e^{-j3\theta} \Rightarrow \angle T(s) = -3\theta$$

$\theta = 90$ $-3\theta = -270$
$\theta = 60$ $-3\theta = -180$
$\theta = 30$ $-3\theta = -90$
$\theta = 0$ $-3\theta = 0$



Page 8



if $-1 < -\frac{K}{8}$ $N=0$ $\rightarrow$ $\frac{K}{8} < 1$ $K < 8$

$N = Z - P$
$0 = Z - 0$ $Z = 0$ // (stable)

if $-\frac{K}{8} < -1$ $N = 2$ $\rightarrow$ $1 < \frac{K}{8}$ $8 < K$

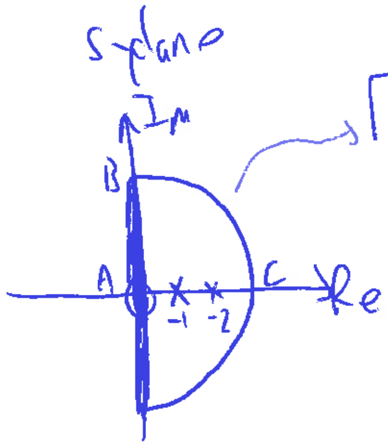
$N = Z - P$
$2 = Z - 0$ $Z = 2$ (unstable)

Page 9

Q7 (c) $T(s) = \frac{Ks}{(s+1)(s-2)}$

$p = 2$ (number of open right half plane poles of $T(s)$)

$p_1 = 1, p_2 = 2$



$\Gamma = \begin{cases} AB, & s = j\omega, \ \omega: 0 \to \infty \\ BC, & s = Re^{j\theta}, \ \theta: 90 \to 0, \ R \approx \infty \end{cases}$

at AB, $s = j\omega$, $T(s) = \frac{Kj\omega}{(j\omega - 1)(j\omega - 2)} = \frac{Kj\omega}{(1 - j\omega)(2 - j\omega)} = \frac{Kj\omega[1+j\omega](2+j\omega)}{[1+\omega^2][4+\omega^2]}$

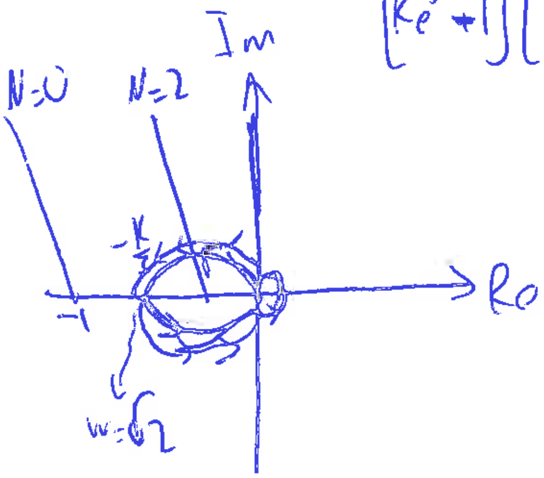
$(1+j\omega)$ $(2+j\omega)$

$T(s) = \frac{Kj\omega[2 - \omega^2 + 3j\omega]}{[1+\omega^2][4+\omega^2]} = jK\frac{\omega[2-\omega^2]}{[1+\omega^2][4+\omega^2]} - K\frac{3\omega^2}{[1+\omega^2][4+\omega^2]}$

	$\omega = 0$	$0 < \omega < \sqrt{2}$	$\sqrt{2} = \omega$	$\omega > \sqrt{2}$	$\omega \to \infty$
$Re(T(s))$	0	$-$	$-\frac{K}{3}$	$-$	0
$Im(T(s))$	0	+	0	$-$	0

at BC, $s = Re^{j\theta}$, $T(s) = \frac{KRe^{j\theta}}{[Re^{j\theta}+1][Re^{j\theta}-2]} = \frac{KRe^{j\theta}}{R^2e^{j2\theta}} = \frac{1}{Re^{j\theta}} = re^{-j\theta}$, $r \simeq 0$

$\angle T(j\omega) = -\theta$



$N = Z - P$

if $-1 < -\frac{K}{3} \Rightarrow \frac{K}{3} < 1$ $K < 3$ $N = 0$ $N = Z - P$

$0 = Z - 2$ $Z = 2$ (unstable)

if $-1 > -\frac{K}{3} \Rightarrow K > 3$ $N = -2$ $N = Z - P$

$-2 = Z - 2$ $Z = 0$ (stable)

Question 3

(a) $T(s) = \frac{1}{s(s+10)}$ $\qquad T(j\omega) = \frac{1}{j\omega(j\omega+10)}$

$|T(j\omega)| = \frac{1}{\omega\sqrt{\omega^2+10^2}}$ $\qquad \angle T(j\omega) = -90 - \arctan\frac{\omega}{10}$

↓ magnitude

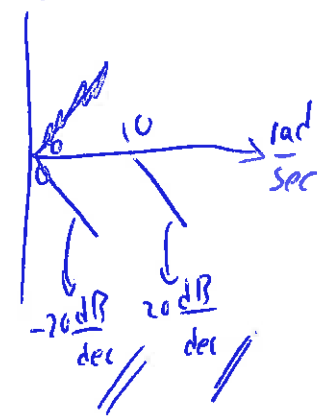
Magnitude

$|T(j\omega)|_{dB} = 20\log\frac{1}{\omega\sqrt{\omega^2+10^2}} = -20\log\omega - 20\log\sqrt{\omega^2+10^2}$

↓ magnitude in dB

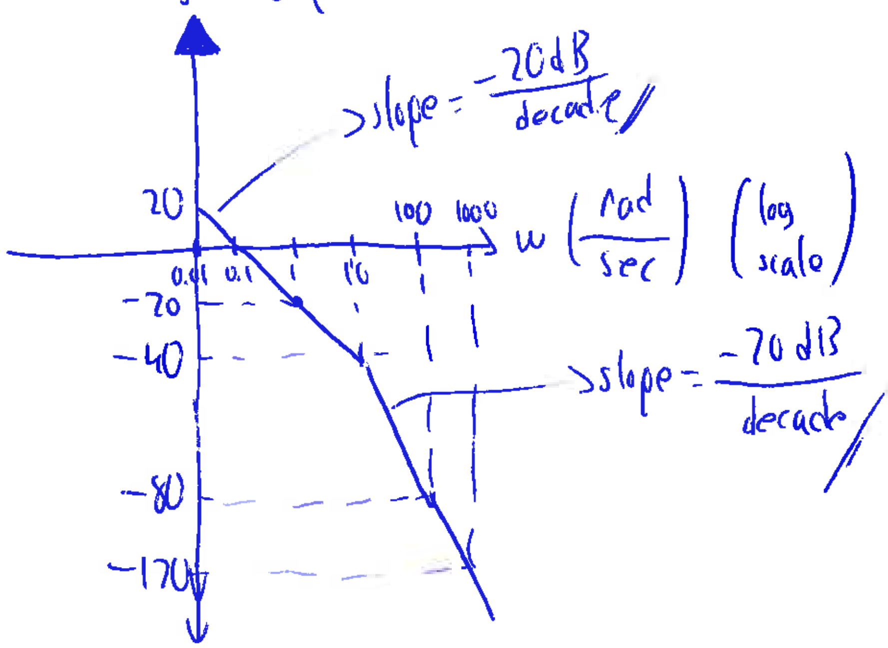
asymptotes $\omega = 0\ \frac{rad}{sec}$ $\qquad \omega = 10\ \frac{rad}{sec}$

effect



$0 < \omega < 10 \Rightarrow |T(j\omega)|_{dB} = -20\log\omega - 20\log 10$
$= -20\log\omega - 20\log 10$

$10 < \omega \Rightarrow |T(j\omega)|_{dB} = -20\log\omega - 20\log\omega = -40\log\omega$



Phase

$\angle T(j\omega) = -90 - \arctan\frac{\omega}{10}$ → **asymptotes**: $\omega_1 = 10 \ \frac{rad}{sec}$

effect:

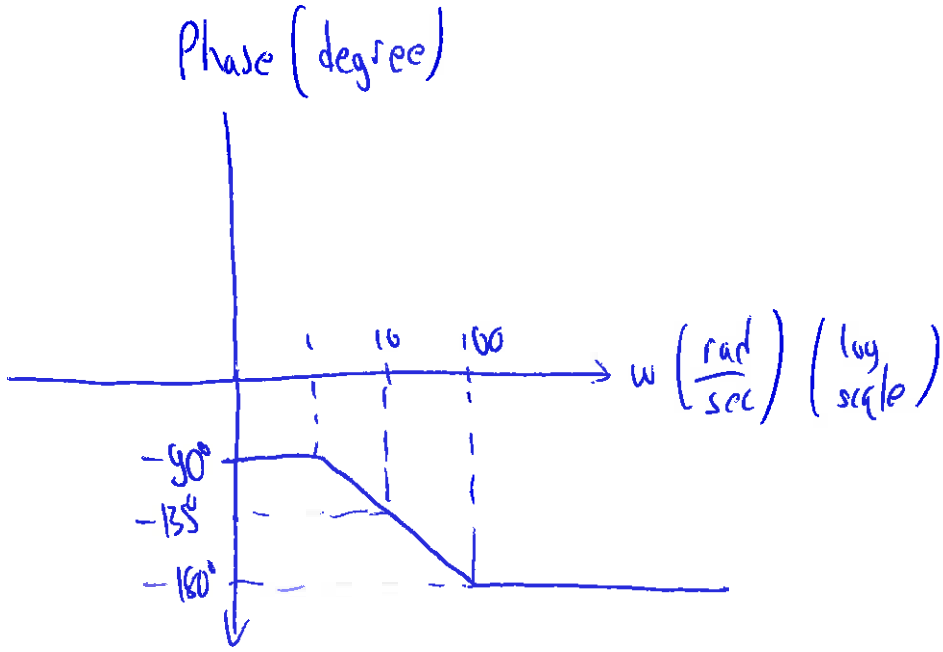
- → $\frac{\omega_1}{10} = 1 \Rightarrow \frac{-45^\circ}{decade}$
- → $10\omega_1 = 100 \Rightarrow \frac{45^\circ}{decade}$

$0 < \omega < \frac{\omega_1}{10} = 1 \Rightarrow \angle T(j\omega) = -90^\circ$

$1 = \frac{\omega_1}{10} < \omega < 10\omega_1 = 100 \Rightarrow \angle T(j\omega) = -90 - 45\log\frac{\omega}{\frac{\omega_1}{10}} = -90 - 45\log$,

$\boxed{\text{if } \omega = 100, \ \angle T(j\omega) = -180^\circ}$

$100 = 10\omega_1 < \omega \Rightarrow \angle T(j\omega) = -180^\circ$



Question 3

Page 12

(b) $T(s) = \dfrac{1}{(s+1)^3}$ $\quad T(j\omega) = \dfrac{1}{[j\omega+1]^3}$ $\quad |T(j\omega)| = \dfrac{1}{[\omega^2+1]^{\frac{3}{2}}}$

$\angle T(j\omega) = -3\arctan\omega$

Magnitude

$$|T(j\omega)|_{dB} = 20\log\frac{1}{[\omega^2+1]^{\frac{3}{2}}} = -20\log[\omega^2+1]^{\frac{3}{2}} = -30\log[\omega^2+1]$$

asymptotes

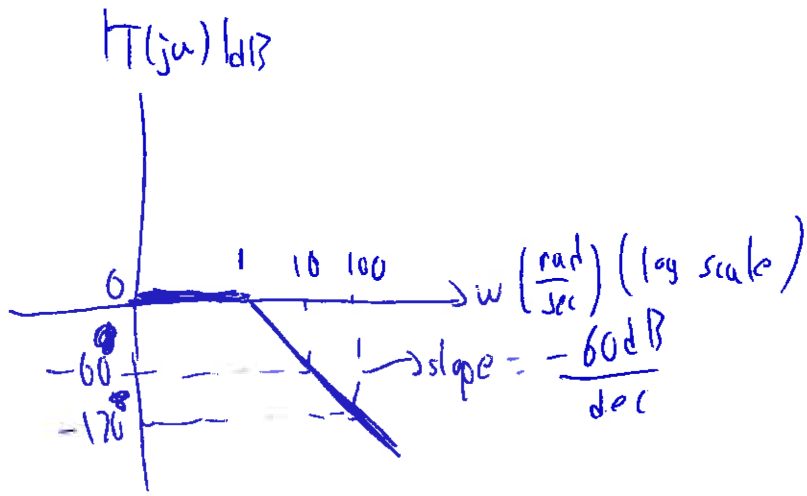
$\omega = 1$

effects

$3\times\left(\dfrac{20dB}{dec}\right) = \dfrac{-60dB}{dec}$

$\omega < 1 \Rightarrow |T(j\omega)|_{dB} \cong -30\log 1 = 0$

$\omega > 1 \Rightarrow |T(j\omega)|_{dB} \cong -30\log\omega^2 = -60\log\omega$



Phase

$\angle T(j\omega) = -3\arctan\omega \rightarrow \omega_1 = 1\,\dfrac{rad}{sec}$ asymptotes

effects

$\dfrac{\omega_1}{10} = 0.1\,\dfrac{rad}{sec} \rightarrow (+3)\left(-\dfrac{45^0}{dec}\right) = \dfrac{-135^0}{dec}$

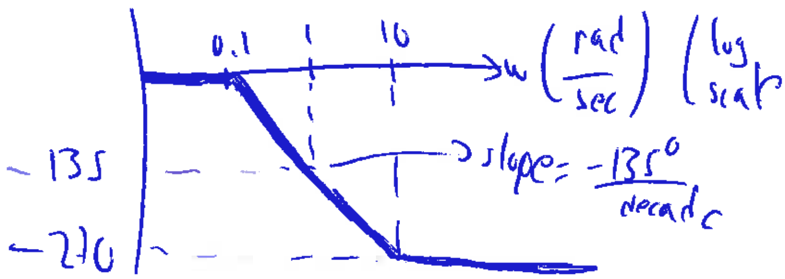
$10\omega_1 = 10\,\dfrac{rad}{sec} \rightarrow (3)\left(+\dfrac{45^0}{dec}\right) = \dfrac{135^0}{dec}$

$0.1 = \frac{\omega_1}{10} < \omega \rightarrow \angle T(j\omega) = 0^\circ$

$0.1 = \frac{\omega_1}{10} < \omega < 10\omega_1 = 10 \Rightarrow \angle T(j\omega) = -135 \log \frac{\omega}{0.1}$

$10 = 10\omega_1 < \omega \Rightarrow \angle T(j\omega) = -270^\circ$ (constant)

$\angle T(j\omega)$ (degree)



Question 3

(c) $T(j\omega) = \dfrac{250 j\omega}{(j\omega+5)(j\omega+50)}$ $\qquad |T(j\omega)| = \dfrac{250\omega}{\sqrt{\omega^2+5^2}\sqrt{\omega^2+50^2}}$

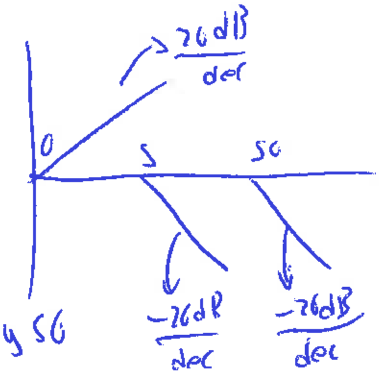
$\angle T(j\omega) = 90 - \arctan\dfrac{\omega}{5} - \arctan\dfrac{\omega}{50}$

Magnitude

$|T(j\omega)|_{dB} = 20\log 250 + 20\log\omega - 20\log\sqrt{\omega^2+5^2} - 20\log\sqrt{\omega^2+50^2}$

asymptotes

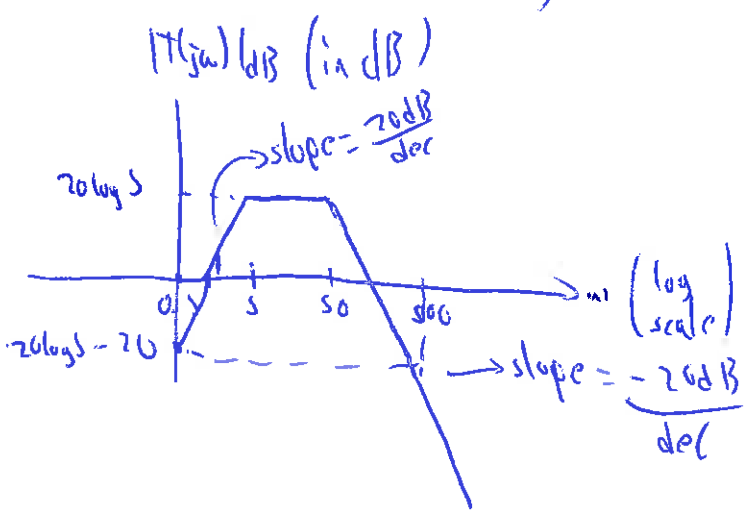
	effects
$\omega_1 = 0 \longrightarrow$	$20\frac{dB}{dec}$
$\omega_2 = 5 \longrightarrow$	$-20\frac{dB}{dec}$
$\omega_3 = 50 \longrightarrow$	$-20\frac{dB}{dec}$



$0 < \omega < 5 \quad |T(j\omega)|_{dB} \simeq 20\log 250 + 20\log\omega - 20\log 5 - 20\log 50$
$= 20\log\omega$

$5 < \omega < 50 \quad |T(j\omega)|_{dB} = 20\log 250 + 20\log\omega - 20\log\omega - 20\log 50$
$= 20\log 5$

$50 < \omega \quad |T(j\omega)|_{dB} = 20\log 250 + 20\log\omega - 20\log\omega - 20\log\omega =$
$= 20\log 250 - 20\log\omega$



Page 15

Phase

effects

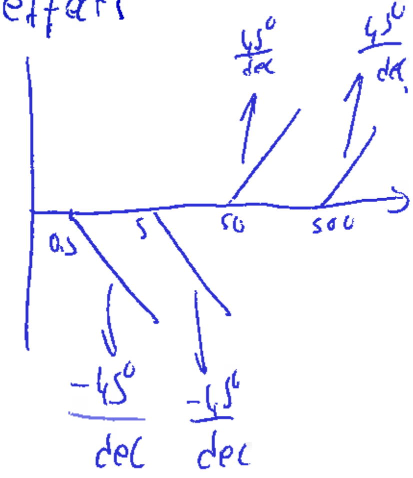
asymptotes → $\omega_1 = 5$ → $0.5 = \frac{\omega_1}{10} \rightarrow \frac{-45^\circ}{dec}$

asymptotes → $\omega_1 = 5$ → $50 = 10\omega_1 \rightarrow \frac{45^\circ}{dec}$

asymptotes → $\omega_2 = 50$ → $5 = \frac{\omega_2}{10} \rightarrow \frac{-45^\circ}{dec}$

asymptotes → $\omega_2 = 50$ → $500 = 10\omega_2 \rightarrow \frac{45^\circ}{dec}$

effects



* $\omega < \frac{\omega_1}{10} = 0.5 \Rightarrow \angle H(j\omega) = 90$ //

* $\frac{\omega_1}{10} = 0.5 < \omega < 5 = \frac{\omega_2}{10} \Rightarrow \angle H(j\omega) = 90 - 45\log\frac{\omega}{0.5}$ //

* $\frac{\omega_2}{10} = 5 < \omega < 50 = 10\omega_1 \Rightarrow \angle H(j\omega) = 45^\circ - 90\log\frac{\omega}{5}$

* $10\omega_1 = 50 < \omega < 500 = 10\omega_2 \Rightarrow \angle H(j\omega) = -45 - 45\log\frac{\omega}{50}$

* $10\omega_2 = 500 < \omega \Rightarrow \angle H(j\omega) = -90^\circ$ //

if $\omega = 5 \Rightarrow \angle H(j\omega) = 90 - 45 = 45^\circ$ //

if $\omega = 50 \rightarrow \angle H(j\omega) = 45 - 90 = -45^\circ$ //

if $\omega = 500 \Rightarrow \angle H(j\omega) = -45 - 45 = -90^\circ$

$\angle H(j\omega)$ (degree)