2013-2016

ECE 386 Homework 2

Q1) $H(s) = K \dfrac{s+2}{s(s-2)} = \dfrac{N(s)}{D(s)}$ $\quad 1 + H(s) = 1 + K \dfrac{s+2}{s^2-2s} = \dfrac{s^2-2s+K(s+2)}{s^2-2s}$

$m = 1$ (degree of numarator)
$n = 2$ (degree of denominator)

* num_branches = max(m,n) = 2
* starting-points = $p_1 = 0 \quad p_2 = 2$
* ending-points: $z_1 = -2$, $z_2 = \infty$
* num_asymptotes = $|n-m| = 1$
* angle-of-asymptotes = $180°$
* Centroid $= \dfrac{\sum_{i=1}^{n} p_i - \sum_{i=1}^{m} z_i}{|n-m|} = \dfrac{(0+2)-(-2)}{1} = 4$

* Break-in, breakaway points

$$\left[\frac{d}{ds} N(s)\right] D(s) - \left[\frac{d}{ds} D(s)\right] N(s) = 0 \Rightarrow (1)s(s-2) - (s+2)(2s-2) = 0$$

$$s^2 - 2s - 2s^2 - 2s + 4 = 0 \qquad -s^2 - 4s + 4 = 0 \qquad \boxed{s^2 + 4s - 4 = 0}$$

$$s_{1,2} = \frac{-4 \mp \sqrt{4^2 - 4 \times (-4) \times 1}}{2} = \frac{-4 \mp \sqrt{32}}{2} = \frac{-4 \mp 4\sqrt{2}}{2} = -2 \mp 2\sqrt{2}$$

(candidate break-in, break-away points)

if $s = -2 + 2\sqrt{2} \Rightarrow s^2 - 2s + K(s+2) = 0$

$$(-2+2\sqrt{2})^2 - 2(-2+2\sqrt{2}) + K(-2+2\sqrt{2}+2) = 0$$

$$4 + 8 - 8\sqrt{2} + 4 - 4\sqrt{2} + K\,2\sqrt{2} = 0$$

$$(16 - 12\sqrt{2}) + K\,2\sqrt{2} = 0 \qquad K = \frac{12\sqrt{2} - 16}{2\sqrt{2}} = 6 - 4\sqrt{2} > 0$$

in root locus

if $\quad s = -2 - 2\sqrt{2} \qquad s^2 - 2s + K(s+2) = 0$

$$(-2-2\sqrt{2})^2 - 2(-2-2\sqrt{2}) + K(-2-2\sqrt{2}+2) = 0$$

$$8 + 4 + 8\sqrt{2} + 4 + 4\sqrt{2} - K\,2\sqrt{2} = 0$$

$$16 + 12\sqrt{2} = 2\sqrt{2}K \qquad K = 6 + 4\sqrt{2} > 0$$

in root locus

* Imaginary axis crossing put $s = j\omega$

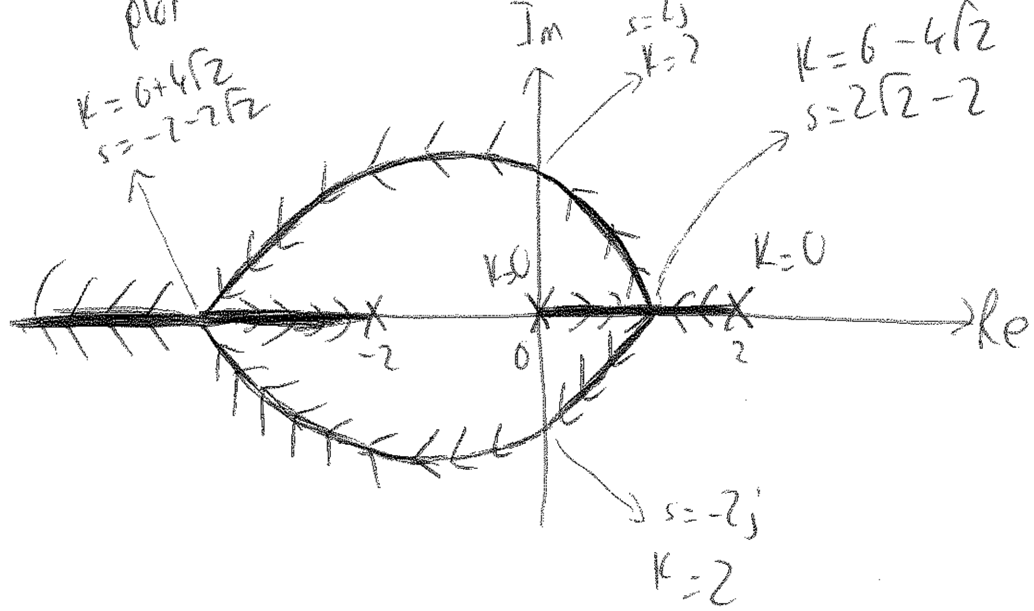
$$s^2 - 2s + K(s+2) = 0$$

$$-\omega^2 - 2j\omega + Kj\omega + 2K = 0 \qquad (2K - \omega^2) + j\omega(K-2) = 0$$

$\omega = 0 \quad K = 0 \Rightarrow K = 0 \Rightarrow s = 0$

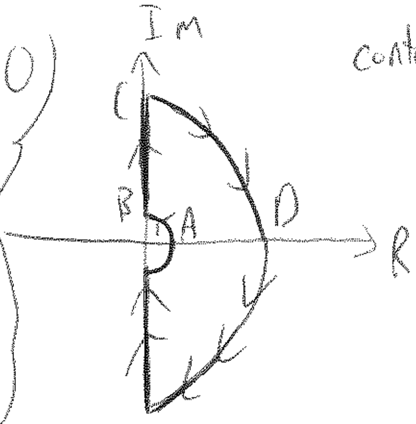
$\omega = \mp 2 \quad K = 2 \Rightarrow K = 2 \Rightarrow s = \mp 2j$

Root-locus plot



Q2)

$$\Gamma = \begin{cases} AB: s = re^{j\theta}, \theta: 0^\circ \to 90^\circ, r \approx 0 \\ BC: s = jw, w: 0 \to \infty \\ CD: s = Re^{j\theta}, \theta: 90^\circ \to 0^\circ, R \approx \infty \\ \text{The remaining part of the contour is symmetric} \end{cases}$$



contour Γ

$$H(s) = K\frac{s+2}{s(s-2)}$$

poles of $H(s) = s=0,\ s=+2$ (singular points)

$\boxed{\text{using the arc } AB \text{ we can move around the open loop pole } s=0}$

$p = \#$ of singular points of $H(s)$ inside contour Γ

(only $s=2$ is in the contour)

Hence $\boxed{p=1}$

* $AB \Rightarrow s = re^{j\theta} \Rightarrow H(s) = K\dfrac{re^{j\theta}+2}{re^{j\theta}(re^{j\theta}-2)} \cong K\dfrac{2}{re^{j\theta}(-2)} = \dfrac{-K}{re^{j\theta}}$

$$\cong \underbrace{R}_{\text{very big}} e^{j(180-\theta)}$$

$\theta = 0^\circ \Rightarrow 180-\theta = 180^\circ$

$\theta = 45^\circ \to 180-\theta = 135^\circ$

$\theta = 90^\circ \to 180-\theta = 90^\circ$

* $BC \Rightarrow s = jw \qquad H(s) = K\dfrac{jw+2}{jw(jw-2)}$ $\boxed{j(jw+2)}$

$$H(s) = K\frac{j(jw+2)(jw+2)}{(-1)w(-4-w^2)} = K\frac{j(4-w^2+4jw)}{w(w^2+4)}$$

Q2 (continue)

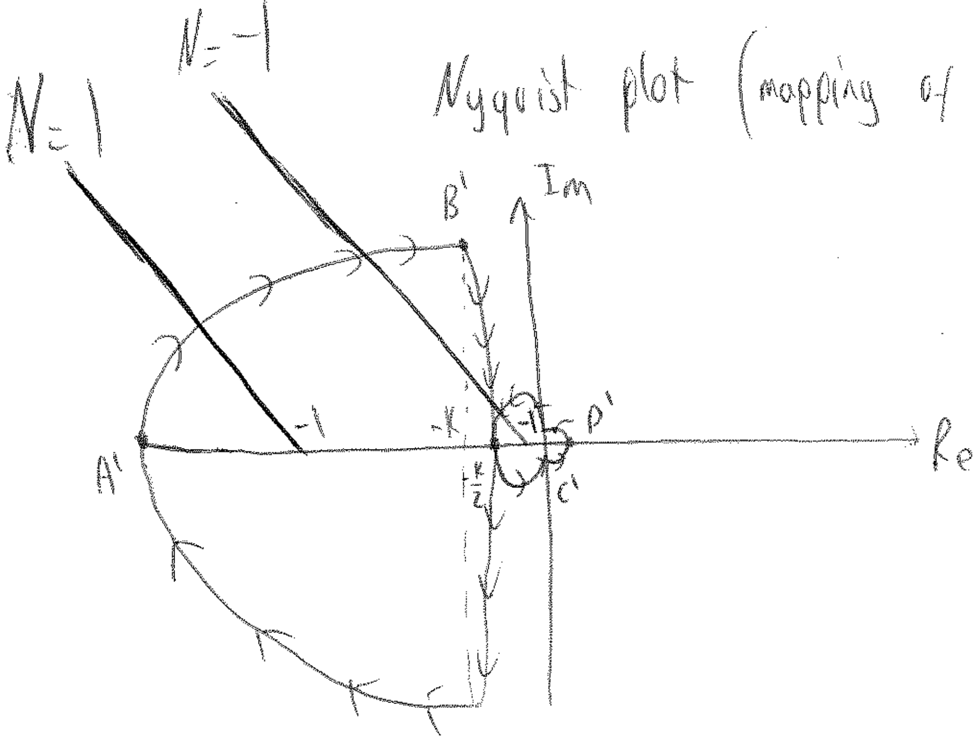
$$H(s) = K\left[j \underbrace{\frac{4-w^2}{w(w^2+4)}}_{Im} - \underbrace{\frac{4}{w^2+4}}_{Re} \right]$$

w	0^+	$0<w<2$	2	$2<w$	$w\to\infty$
$Im(H(s))$	$+\infty$	$+$	0	$-$	0
$Re(H(s))$	$-K$	$-$	$-\frac{K}{2}$	$-$	0

$$*\ (D) \Rightarrow s = Re^{j\theta} \quad H(s) = K\,\frac{Re^{j\theta}+2}{Re^{j\theta}(Re^{j\theta}-2)} \cong K\,\frac{Re^{j\theta}}{R^2e^{2j\theta}} = \underset{\text{very small}}{r}\, e^{-j\theta}$$

$$\theta = 90 \Rightarrow -\theta = -90$$
$$\theta = 45 \Rightarrow -\theta = -45$$
$$\theta = 0 \Rightarrow -\theta = 0$$

Nyquist plot (mapping of $H(s)$ over the contour Γ)



N = # of encirclement of -1 by the Nyquist plot in clockwise direction

Q2) continue

if $-1 < -\frac{K}{2} \rightarrow N = +1$ $\quad N = z - p \quad 1 = z - p \Rightarrow z = 2$ (unstable system)

$K < 2$

z : # of closed loop poles of $H(s)$ [zeros of $1 + H(s)$] in open right half plane

if $-\frac{K}{2} < -1 \quad N = -1$ $\quad N = z - p \quad -1 = z - 1 \Rightarrow z = 0$ (stable system)

$2 < K$

Result
$$\begin{cases} \text{if } K < 2 \quad z = 2 \text{ (two unstable closed loop poles)} \rightarrow \text{unstable} \\ \text{if } K > 2 \quad z = 0 \text{ (0 unstable closed loop poles)} \rightarrow \text{stable} \end{cases}$$

Q3) $H(s) = \frac{\sqrt{20}}{s(s+1)}$ $\quad s = j\omega \quad H(j\omega) = \frac{\sqrt{20}}{j\omega(j\omega+1)} \quad |H(j\omega)| = \frac{\sqrt{20}}{\omega\sqrt{\omega^2+1}}$

(a) at gain crossover frequency ω_g $\quad |H(j\omega_g)| = 1 = \frac{\sqrt{20}}{\omega_g\sqrt{\omega_g^2+1}}$

$\omega_g^2(\omega_g^2+1) = 20 \quad \omega_g^2 = x \quad x(x+1) = 20 \quad x = 4 \quad \omega_g = 2$

(b) $\phi_{margin} = 180 + \angle H(j\omega_g)$ $\quad \angle H(j\omega_g) = -90 - \tan^{-1}\frac{2}{1}$

$\angle H(j\omega) = -90 - \tan^{-1}\frac{\omega}{1}$ $\quad \phi_{margin} = 180 + (-90 - \tan^{-1} 2)$

$\phi_{margin} = 90 - \tan^{-1} 2$

Q4) $H(s) = \dfrac{10}{s(s+1)^2}$, $H(j\omega) = \dfrac{10}{j\omega(j\omega+1)^2}$, $\angle H(j\omega) = -90 - 2\,\mathrm{Tan}^{-1}\omega$

(a) At $\omega = \omega_c$ (phase crossover frequency) $\angle H(j\omega_c) = -180 = -90 - 2\,\mathrm{Tan}^{-1}\omega_c$

$2\,\mathrm{Tan}^{-1}\omega_c = 90 \quad \mathrm{Tan}^{-1}\omega_c = 45 \quad \omega_c = 1\,\dfrac{rad}{sec}$

(b) Gain_margin $= G_M = \dfrac{1}{|H(j\omega_c)|} = \dfrac{1}{5} = \dfrac{1}{5}$

$|H(j\omega)| = \dfrac{10}{\omega\,(\omega^2+1)} \Rightarrow |H(j\omega_c)| = \dfrac{10}{1\,(1^2+1)} = \dfrac{10}{2} = 5$

Gain_margin $= \dfrac{1}{5}$