ECE 488 – Automatic Control

Bode Plot

Assistant Prof. Dr. Klaus Schmidt

Department of Mechatronics Engineeering – Çankaya University

Compulsory Course in Electronic and Communication Engineering
Credits (3/0/3)

Course Webpage: http://ECE488.cankaya.edu.tr

Reminder

Previous Weeks

- LTI system modeling
- Nonlinear modeling and linearization
- Stability
- Steady-state and transient response
- Feedback Control
  - Root locus
  - Nyquist plot

This week

- Frequency response
- Bode plot

Frequency Response: Basic Idea

Given

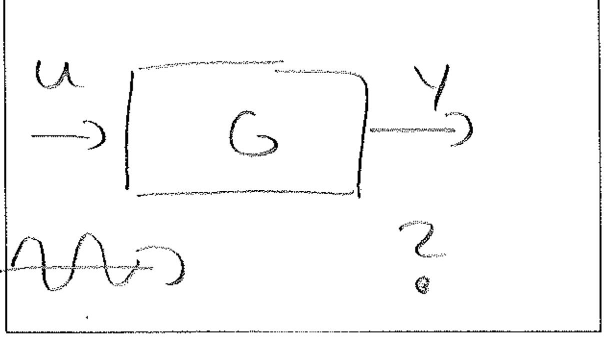
- Stable LTI system with transfer function $G(s)$

Goal

- Find system response $y(t)$ for sinusoidal input signal

$$u(t) = \sin(\omega t)$$

Gap 1



Solution

- Consider output computation in the Laplace domain: $Y(s) = G(s)\,U(s)$
- Sinusoidal input function: $U(s) = \dfrac{\omega}{s^2 + \omega^2}$

Frequency Response: Output Computation

Output Computation

Gap 2

$$Y(s) = G(s)\cdot U(s) = G(s)\cdot\frac{\omega}{(s+j\omega)(s-j\omega)} =$$

Residues for steady-state response:

$$p_1 = -j\omega,\quad r_1 = \lim_{s\to -j\omega} \frac{G(s)\cdot\omega}{s-j\omega} = \frac{G(-j\omega)\cdot\omega}{-2j\omega}$$

$$p_2 = +j\omega,\quad r_2 = \lim_{s\to j\omega} \frac{G(s)\cdot\omega}{s+j\omega} = \frac{G(j\omega)\cdot\omega}{2j\omega}$$

$$\Rightarrow Y_{ss}(s) = \frac{r_1}{s+j\omega} + \frac{r_2}{s-j\omega} =$$

$$= -\frac{G(-j\omega)}{2j}\cdot\frac{1}{s-j\omega} + \frac{G(j\omega)}{2j}\,\frac{1}{s-j\omega}$$

Frequency Response: Steady State Response

Intermediate Result

$$Y_{ss}(s) = \frac{G(j\omega)}{2j}\frac{1}{s-j\omega} + \frac{G(-j\omega)}{-2j}\frac{1}{s+j\omega}$$

Computation of the Steady State Response

Gap 3

Write using magnitude and phase:

$$Y_{ss}(s) = \frac{1}{2j}\left(\frac{|G(j\omega)|\, e^{j\angle G(j\omega)}}{s-j\omega} - \frac{|G(j\omega)|\, e^{-j\angle G(j\omega)}}{s+j\omega}\right) =$$

$$\Rightarrow y_{ss}(t) = |G(j\omega)| \left(\frac{e^{j(\omega t+\angle G(j\omega))} - e^{-j(\omega t+\angle G(j\omega))}}{2j}\right) =$$

$$= |G(j\omega)| \sin(\omega t + \angle G(j\omega))$$

Frequency Response: Result

Result

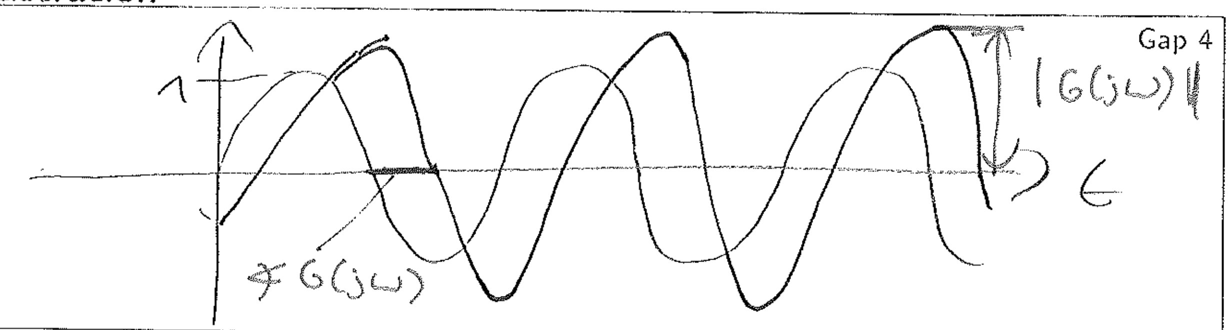
$$y(t) = |G(j\omega)| \sin\left(\omega t + \angle(G(j\omega))\right)$$

Description

- Output signal y oscillates with same frequency ω as input signal u
- Amplification of u by $|G(j\omega)|$
- Phase shift of u by $\angle(G(j\omega))$

Illustration

Gap 4



Bode Plot: Basic Idea

Description

- Given: Transfer function $G(s)$
- Task: Show the frequency response in terms of magnitude $|G(j\omega)|$ and phase shift $\angle(G(j\omega))$

Magnitude Plot

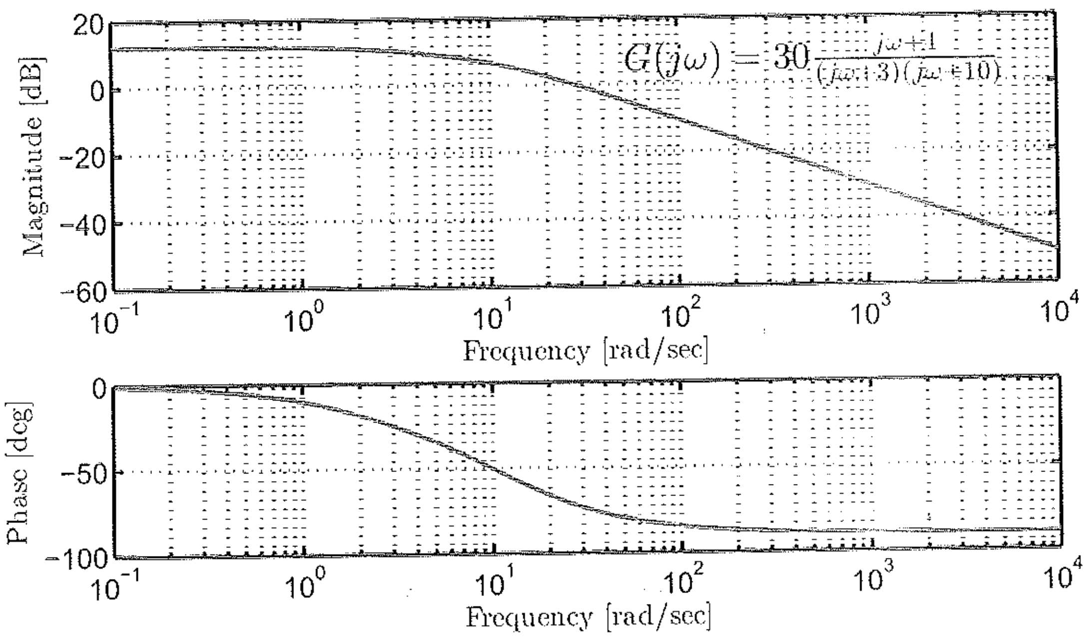
- Frequency axis with logarithmic scale ω [rad/sec]
- Magnitude axis with $20 \log|G(j\omega)|$ [dB]

Phase Plot

- Frequency axis with logarithmic scale ω [rad/sec]
- Phase axis with $\angle G(j\omega) = \arctan(\frac{\mathrm{Im}(G(j\omega))}{\mathrm{Re}(G(j\omega))})$ [°]

Bode Plot: Example

Bode Plot Example



Bode Plot: Transfer Function Representation

Time-constant Representation

$$G(s) = K_{\mathrm{DC}} \frac{(1+\tau_1 s)(1+\tau_2 s)\cdots(1+2\delta_f \tau_f s + \tau_f^2 s^2)\cdots}{s^q (1+T_1 s)(1+T_2 s)\cdots(1+2D_g T_g s + T_g^2 s^2)\cdots}$$

- Time constants for real zeros/poles: $\tau_1, \tau_2, \cdots;\ T_1, T_2, \cdots$
- Time constants for conjugated complex zeros/poles: $\tau_f, \cdots;\ T_g, \cdots$
- Damping factor for conjugated complex zeros/poles: $\delta_f, \cdots;\ D_g, \cdots$
- Multiplicity of pole at zero: q
  $\Rightarrow$ If the transfer function is not given in the time-constant representation, it has to be transformed to this representation

Frequency Response

$$G(s) = K_{\mathrm{DC}} \frac{(1+j\omega\tau_1)(1+j\omega\tau_2)\cdots(1+j\omega 2\delta_f \tau_f + (j\omega)^2 \tau_f^2)\cdots}{(j\omega)^q (1+j\omega T_1)(1+j\omega T_2)\cdots(1+j\omega 2 D_g T_g + (j\omega)^2 T_g^2)\cdots}$$

Bode Plot: Transfer Function Representation

Example

Gap 5

$$G(s) = \frac{0.1 + 0.4 s}{10 s (0.1 + s + 10 s^2)}$$ $\rightarrow$ compute time-constant representation

$$G(s) = \frac{0.1(1+4s)}{s(1+10s+100s^2)}$$

$\rightsquigarrow q = 1;\ K_{DC} = 0.1;\ \tau_1 = 4;\ T_1 = 10;\ D = \frac{1}{2}$

Standard Numerator/Denominator Factors

- K_{DC}
- s
- $1 + T s$
- $1 + 2 D T s + T^2 s^2$

Bode Plot: Common Examples

DC Gain: $G(j\omega) = K_{DC}$

- Magnitude: $|G(j\omega)| = |K_{DC}| \Rightarrow |G(j\omega)|_{dB} = 20 \log K_{DC}$
- Phase $\angle(G(j\omega)) = \begin{cases} 0^\circ & \text{if } K_{DC} > 0 \\ 180^\circ & \text{if } K_{DC} < 0 \end{cases}$

Integrator $G(j\omega) = \dfrac{1}{j\omega}$

- Magnitude: $|G(j\omega)| = \dfrac{1}{\omega} \Rightarrow |G(j\omega)|_{dB} = -20 \log \omega$
- Phase: $\angle G(j\omega) = -90^\circ$

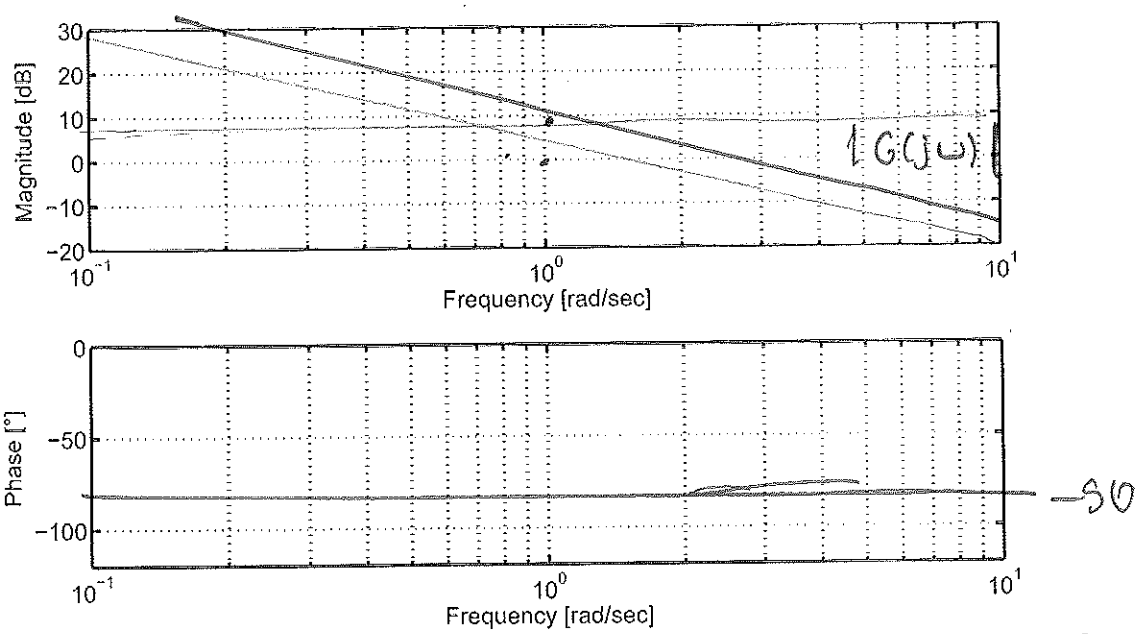
Combination of DC Gain and Integrator: $G(j\omega) = K_{DC}\,\dfrac{1}{j\omega}$

- Magnitude: $G(j\omega)|_{dB} = 20 \log K_{DC} - 20 \log \omega$
- Phase: $\angle(G(j\omega)) = \angle(K_{DC}) - 90^\circ$

Bode Plot: Common Examples

$K_{DC} = 2 \,\hat{=}\, 6\,dB$

Bode Plot Construction: $G(s) = \dfrac{2}{s}$



Bode Plot: Examples

First-order Lag $G(j\omega) = \dfrac{1}{1 + j\omega T}$

- Magnitude: $|G(j\omega)| = \dfrac{1}{\sqrt{1+\omega^2 T^2}}$

$$\Rightarrow |G(j\omega)|_{dB} = 20\big(-1/2 \log(1+\omega^2 T^2)\big)$$

$$\approx \begin{cases} 0 & \omega < 1/T \\ -20 \log \omega T & \omega > 1/T \end{cases}$$

$\Rightarrow$ Straight-line approximation that bends at $\omega = 1/T$

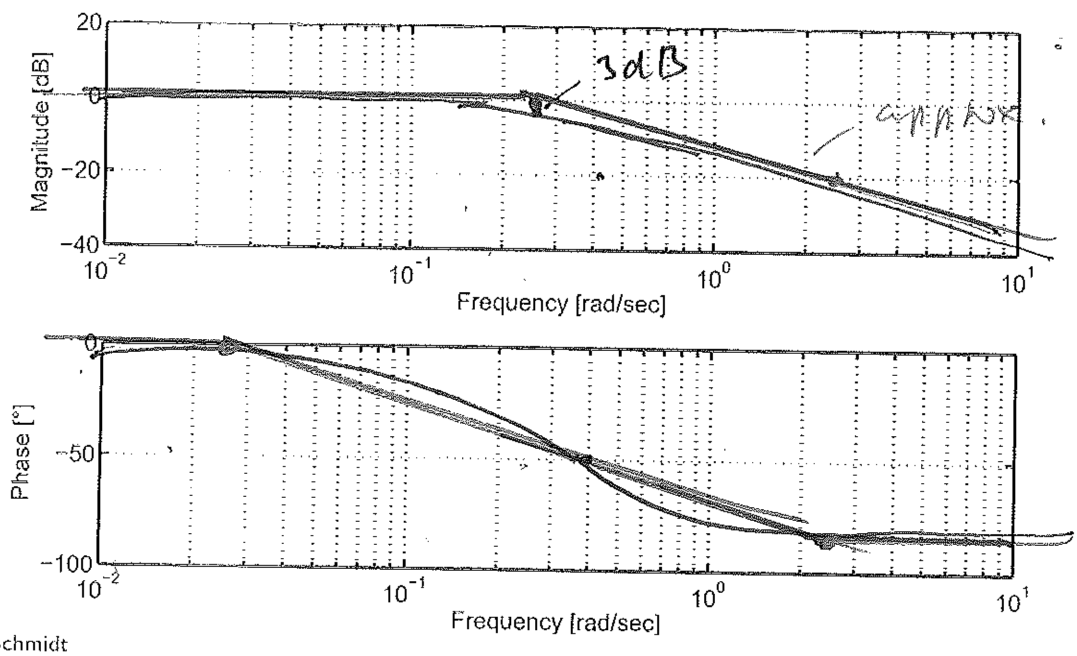
- Phase: $\angle G(j\omega) = -\angle(1 + j\omega T) = -\arctan \omega T$

$$\Rightarrow \angle G(j\omega) \approx \begin{cases} 0^\circ & \omega < 1/(10\,T) \\ -90^\circ & \omega > 10/T \end{cases}$$

$\Rightarrow$ Straight-line approximation that decreases from 0 to -90° between $\omega = 1/(10\,T)$ to $\omega = 10/T$

Bode Plot: First-order Lag

Bode Plot Construction: $G(s) = \dfrac{1}{1+4s}$ $\Rightarrow \frac{1}{T} = 0.25$



Bode Plot: Second-order Lag

Second-order Lag $G(j\omega) = \dfrac{1}{1 + 2DTj\omega + T^2(j\omega)^2}$

- Magnitude: $|G(j\omega)| = \dfrac{1}{\sqrt{(1 - T^2\omega^2)^2 + 4D^2T^2\omega^2}}$

$$\Rightarrow |G(j\omega)|_{dB} = 20\Big(-1/2 \log\left((1 - T^2\omega^2)^2 + 4D^2T^2\omega^2\right)\Big)$$

$$\approx \begin{cases} 0 & \omega < 1/T \\ -40 \log \omega T & \omega > 1/T \end{cases}$$

$\Rightarrow$ Straight-line approximation that bends at $\omega = 1/T$

- Phase: $\angle G(j\omega) = -\angle(j2DT\omega + 1 - \omega^2T^2) = -\arctan \frac{2DT\omega}{1-\omega^2T^2}$

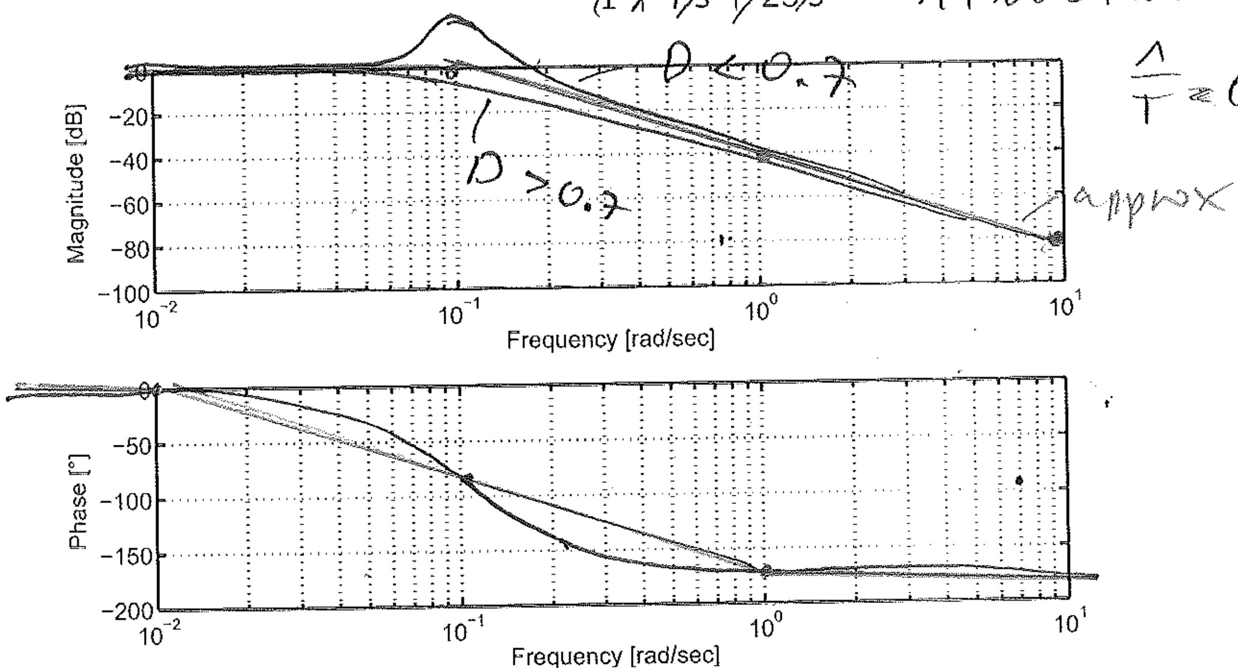
$$\Rightarrow \angle G(j\omega) \approx \begin{cases} 0^\circ & \omega << 1/(10\,T) \\ -180^\circ & \omega >> 10/T \end{cases}$$

$\Rightarrow$ Straight-line approximation that decreases from 0 to -180° between $\omega = 1/(10\,T)$ and $\omega = 10/T$

Bode Plot: Examples

Bode Plot Construction: $G(s) = \cancel{\dfrac{1}{1 + 7s + 25s^2}}$ $\quad \dfrac{1}{1 + 10s + 100s^2}$

$\frac{1}{T} \approx 0.1$



Bode Plot: Multiplication Rules

Multiplication of Transfer Functions

$$G(s) = G_1(s) \cdot G_2(s) \cdot \dots \cdot G_n(s)$$

Addition of Magnitude and Phase in Bode Plot

- $|G(j\omega)|_{dB} = |G_1(j\omega)|_{dB} + |G_2(j\omega)|_{dB} + \dots + |G_n(j\omega)|_{dB}$
- $\angle G(j\omega) = \angle G_1(j\omega) + \angle G_2(j\omega) + \dots + \angle G_n(j\omega)$

Gap 6

$$G(s) = \frac{0.1(1+4s)}{s(1+10s+100s^2)}$$

$$|G(j\omega)|_{dB} \; |0.1|_{dB} + |1+4j\omega|_{dB} - |j\omega|_{dB} - |1+10j\omega - 100\omega^2|$$

$$\sphericalangle G(j\omega) = 0° + \sphericalangle(1+4j\omega) - \sphericalangle(j\omega) - \sphericalangle(1+10j\omega - 100\omega^2)$$

Bode Plot: Transfer Function Zeros

Inverse of Transfer Functions

$$G(s) = G_1^{-1}(s)$$

Negation of Magnitude and Phase

- $|G(j\omega)|_{dB} = -|G_1(j\omega)|_{dB}$
- $\angle G(j\omega) = -\angle G_1(j\omega)$

Gap 7

$$G(s) = (1+4s) \rightsquigarrow G_1(s) = \frac{1}{1+4s}$$

$$|G(j\omega)|_{dB} = -\left|\frac{1}{1+4j\omega}\right|_{dB}$$

$$\sphericalangle G(j\omega) = -\sphericalangle\left(\frac{1}{1+4j\omega}\right)$$

Bode Plot: Example

Computation

Gap 8

$$G(s) = \frac{0.1\,(1+4s)}{s\,(1+10s+100s^2)}$$

$\Rightarrow$ 4 factors | $|0.1|_{dB} = -20\,dB$

$|1+4j\omega|_{dB} = -\left|\frac{1}{1+4j\omega}\right|_{dB}$

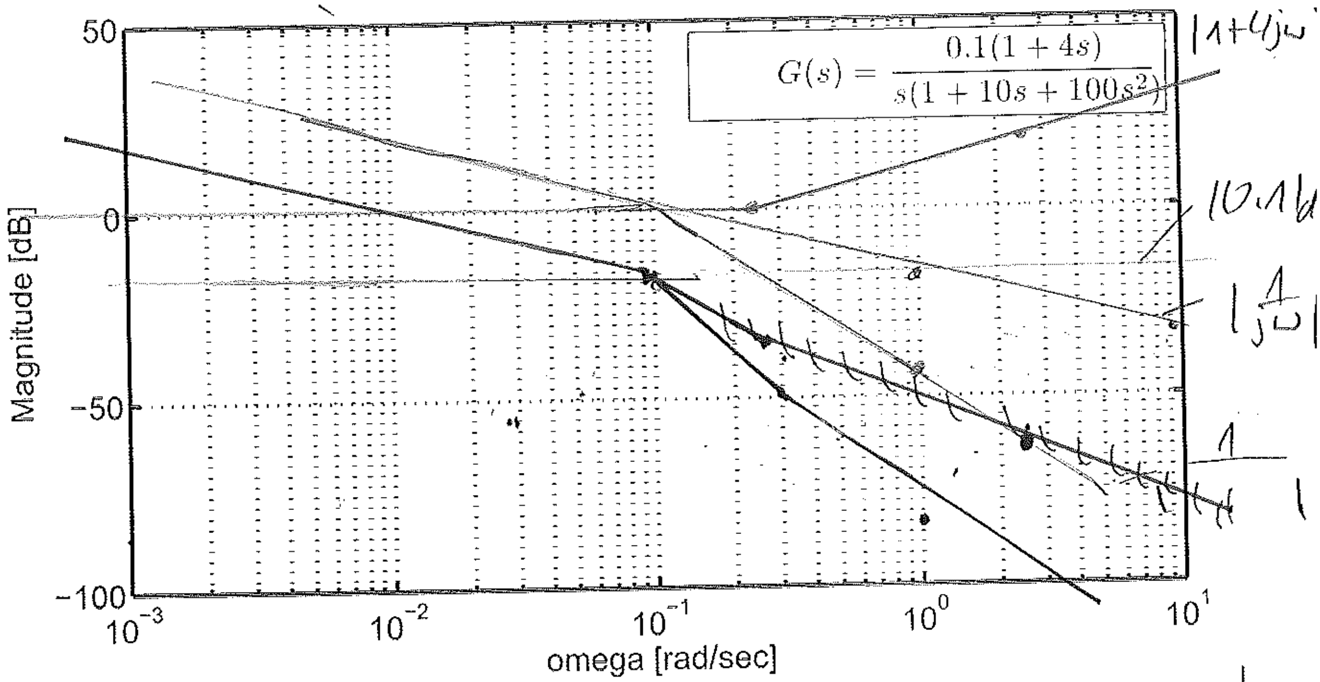
$\left|\frac{1}{j\omega}\right|_{dB}$ see before

$\left|\frac{1}{1+10j\omega-100\omega^2}\right|$ see before

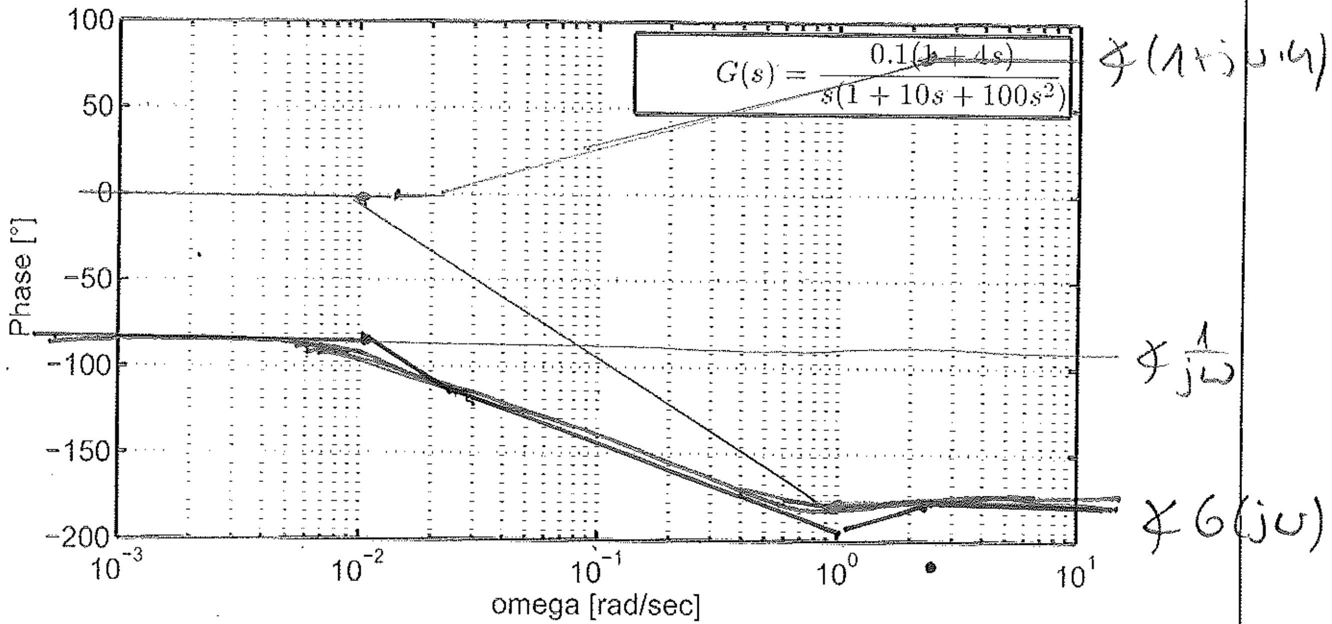
$\measuredangle(1+4j\omega) = -\measuredangle\left(\frac{1}{1+4j\omega}\right)$ $\quad \measuredangle\left(\frac{1}{j\omega}\right) = -90^0$

$\measuredangle\left(\frac{1}{1+10j\omega-\omega^2}\right)$ see before

Bode Plot: Example



Bode Plot: Example



Bode Plot: Non-minimum Phase Factors

First-order Lag Example

$$G(s) = \frac{1}{1 - Ts}$$

Comparison to Minimum-Phase Factor

- $|G(j\omega)|_{dB} = |\frac{1}{1+j\omega T}|_{dB}$; $\angle G(j\omega) = -\angle(\frac{1}{1+j\omega T})$

Gap 9

$$\left|\frac{1}{1-j\omega T}\right| = \frac{1}{(1+\omega^2T^2)^{\frac{1}{2}}} = \left|\frac{1}{1+j\omega T}\right|$$

$$\angle\left(\frac{1}{1-j\omega T}\right) = -\arctan\left(\frac{-\omega T}{1}\right) = +\arctan\left(\frac{\omega T}{1}\right) =$$

$$= -\angle\left(\frac{1}{1+j\omega T}\right)$$

Bode Plot: Example

Computation

Gap 10

$G(s) = \frac{1+0.1s}{1-s}$

2 factors: $|1+0.1j\omega|_{dB} \rightarrow 1/T = 10$

$-|1-j\omega|_{dB} = -|1+j\omega|_{dB} \rightarrow 1/T = 1$

Bode Plot: Example