ECE 488 – Automatic Control

Nyquist Plot

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Compulsory Course in Electronic and Communication Engineering
Credits (3/0/3)

Course Webpage: http://ECE488.cankaya.edu.tr

Reminder

Previous Weeks

- Plant modeling
- Properties of transfer functions
- Feedback loop
  - Sensitivity transfer functions
  - Internal stability
- Root locus method

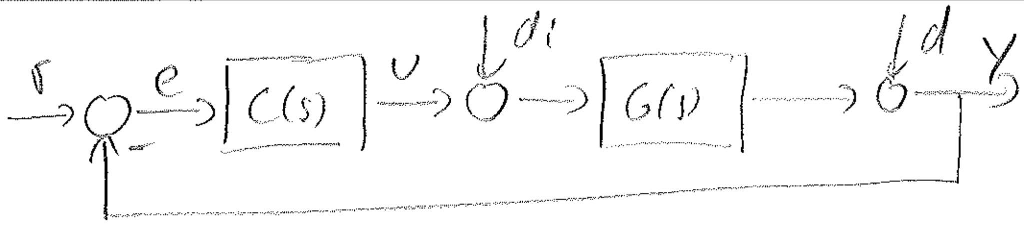
This week

- Nyquist plot
- Nyquist criterion

Stability Analysis: Reminder

Basic Feedback Control Loop

Gap 1



- The feedback loop is internally stable if all zeros of $1 + C(s)\,G(s)$ lie in the OLHP
  - $\Rightarrow$ Direct computation of the zeros of $1 + C(s)\,G(s)$
  - $\Rightarrow$ Analytical verification using Routh-Hurwitz test
  - $\Rightarrow$ Graphical verification based on root locus plot
  - $\Rightarrow$ This week: Graphical verification using Nyquist plot

Nyquist Plot: Frequency Response

Open Loop Properties

$$G_o(s) = C(s) \cdot G(s) = K\,\frac{(s - z_1)\,(s - z_2)\,\cdots\,(s - z_m)}{(s - p_1)\,(s - p_2)\,\cdots\,(s - p_n)}$$

- $G_o(s)$ has poles $p_1, \cdots, p_n$
- $G_o(s)$ is a complex number for all $s \in \mathbb{C}$

Example

Gap 2

$G_o(s) = \frac{1}{s(s+1)}$; Different values of $s \in \mathbb{C}$:

$s = 2j \Rightarrow G_o(2j) = \frac{1}{2j(2j+1)} = \frac{1}{-4+2j} = \frac{-4-2j}{16+4} \in \mathbb{C}$

$s = 1 \Rightarrow G_o(1) = \frac{1}{1(1+1)} = \frac{1}{2} \in \mathbb{C}$

$s = -1+j \Rightarrow G_o(-1+j) = \frac{1}{(-1+j)\cdot j} = \frac{1}{-j-1} = \frac{-1+j}{2} \in \mathbb{C}$

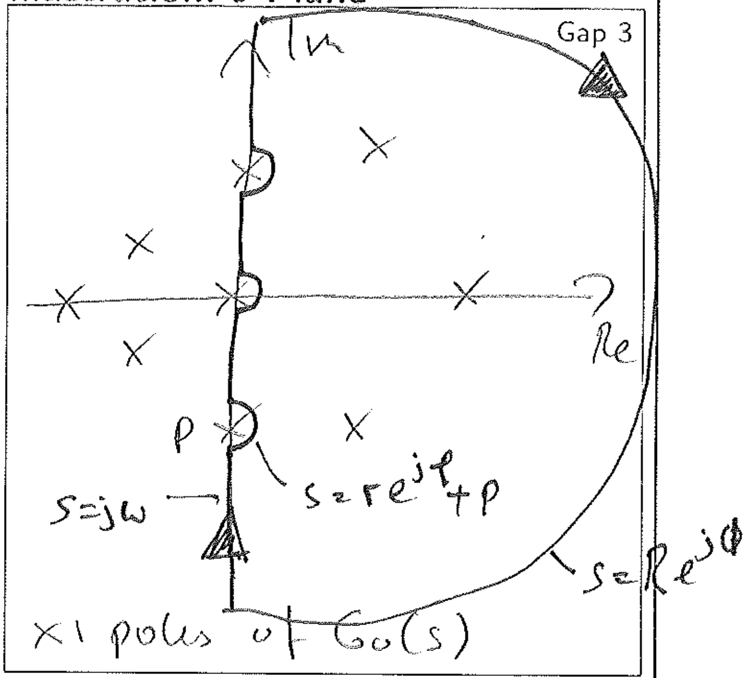
$\Rightarrow$ **Plot $G_o(s)$ in the complex plane along a well-defined path for s**

Nyquist Plot: Construction

Closed Path C in Complex Plane

- C includes the imaginary axis
- C encircles all poles of G_o on the imaginary axis by a small semi-circle in the right-half plane
- C closes by a large semi-circle in the right-half plane
- C is traversed in clockwise direction
  ⇒ **Such path C encircles all poles of $G_o(s)$ in the right-half plane**

Illustration: s-Plane



Nyquist Plot: Path Representation

Components of the Closed Path C

- Imaginary axis: $s = j\omega$ for $-\infty \leq \omega \leq \infty$
- Large semi-circle: $s = R \cdot e^{j\Phi}$ for $\pi/2 \geq \Phi \geq -\pi/2$ and R large
- Small semi-circles for each p_i on imaginary axis: $s = p_i + r \cdot e^{j\varphi}$ for $-\pi/2 \leq \varphi < \pi/2$ and r small

Nyquist Curve

- Defined by mapping $s \mapsto G_o(s)$ for all $s \in C$
  ⇒ Each $s \in C$ is mapped to the complex number $G_o(s)$

Nyquist Plot

- Graphical representation of the Nyquist curve in the complex plane
  → We call the complex plane G_o-plane
  → Complex number $G_o(s)$ is plotted in the G_o-plane for each $s \in C$

Nyquist Plot: Assumptions

Assumptions for $G_o(s)$

$$G_o(s) = K\,\frac{(s-z_1)\cdots(s-z_m)}{s^q(s-p_{q+1})\cdots(s-p_n)}$$

- $G_o(s)$ is proper: $r = n - m \geq 0$
- $G_o(s)$ has $0 \leq q \leq n$ poles at zero and $n-q$ poles different from zero

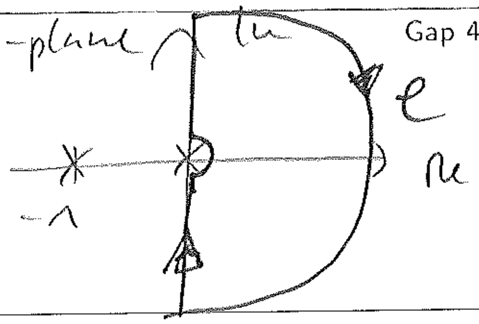
Example

Gap 4

$G_o(s) = \frac{1}{s(s+1)}$

$\Rightarrow q = 1$, pole at $p_2 = -1$

$\Rightarrow r = 2 \geq 0$



Nyquist Plot: Construction

Large Semi-Circle: $s = R \cdot e^{j\Phi}$, $\pi/2 \geq \Phi \geq -\pi/2$

$$\lim_{R\to\infty} G_o(R\cdot e^{j\Phi}) \approx K \cdot \frac{R^m e^{jm\Phi}}{R^n e^{jn\Phi}} = K\cdot\frac{1}{R^r}e^{-jr\Phi} = \begin{cases} 0 & \text{if } r > 0 \\ K & \text{if } r = 0 \end{cases}$$

$\Rightarrow$ Maps to point on real axis in complex plane

Example

Gap 5

$G_o(s) = \frac{1}{s(s+1)}$

For $s = Re^{j\phi}$, $\frac{\pi}{2} \geq \phi \geq -\pi/2$, R large

$G_o(s) = \frac{1}{Re^{j\phi}(Re^{j\phi}+1)} \approx \frac{1}{R^2 e^{-2j\phi}} \approx 0$ for $R \to \infty$

$\Rightarrow G_o(s) \approx 0$ if s moves on large semi-circle

Nyquist Plot: Construction

Small Semi-Circles for poles at zero: $s = r \cdot e^{j\varphi}$, $-\pi/2 \leq \varphi \leq \pi/2$

$$G_o(r \cdot e^{j\varphi}) \approx K \cdot \frac{(-z_1)\cdots(-z_m)}{r^q e^{jq\varphi}(-p_{q+1})\cdots(-p_n)} = K_r \cdot \frac{1}{r^q} \cdot e^{-jq\varphi}$$

⇒ Maps to a curve with large radius that encircles the origin $q/2$ times

~~⇒ Direction of the curve is clockwise if K_r is positive, otherwise counter-clockwise~~ circle starts at $\measuredangle K_r + \frac{q\pi}{2}$, ends at $\measuredangle K_r - \frac{q\pi}{2}$

Example

Gap 6

$G_o(s) = \frac{1}{s(s+1)}$; small semi-circle $re^{j\varphi}$ at $s=0$

$G_o(re^{j\varphi}) = \frac{1}{re^{j\varphi}(1+re^{j\varphi})} \approx \frac{1}{re^{j\varphi}} = \frac{1}{r} e^{-j\varphi}$ for small r

⇒ large semi-circle; start: $\frac{\pi}{2}$, end: $-\pi/2$

⇒ $G_o(s)$ moves on large semi-circle if s moves on $re^{j\varphi}$

Nyquist Plot: Construction

Imaginary Axis: $s = j\omega$, $-\infty < \omega < \infty$

$$G_o(j\omega) = K \frac{(j\omega - z_1)\cdots(j\omega - z_m)}{(j\omega)^q(j\omega - p_{q+1})\cdots(j\omega - p_n)} =$$

$$= K_r \frac{(1 - j\omega/z_1)(1 - j\omega/z_2)\cdots(1 - j\omega/z_m)}{(j\omega)^q(1 - j\omega/p_1)(1 - j\omega/p_2)\cdots(1 - j\omega/p_n)}$$

$$K_r = K \frac{(-z_1)(-z_2)\cdots(-z_m)}{(-p_1)(-p_2)\cdots(-p_n)}$$

Limit for $\omega \to 0$

$$G_o(j\omega) \approx \frac{K_r}{(j\omega)^q}$$

⇒ Phase: $\lim_{\omega \to 0} \measuredangle(G_o(j\omega)) = \measuredangle(K_r) - q\frac{\pi}{2}$

⇒ Magnitude: $\lim_{\omega \to 0} |G_o(j\omega)| = \begin{cases} \infty & \text{if } q > 0 \\ |K_r| & \text{if } q = 0 \end{cases}$

Nyquist Plot: Construction

Imaginary Axis: $s = j\omega$, $-\infty < \omega < \infty$

$$G_o(j\omega) = K\frac{(j\omega - z_1)\cdots(j\omega - z_m)}{(j\omega)^q(j\omega - p_{q+1})\cdots(j\omega - p_n)}$$

Limit for $\omega \to \infty$

$$G_o(j\omega) \approx K\frac{(j\omega)^m}{(j\omega)^n} = \frac{K}{(j\omega)^{n-m}} = \frac{K}{(j\omega)^r}$$

$\Rightarrow$ Phase: $\lim_{\omega\to\infty} \angle(G_o(j\omega)) = \angle(K) - r\frac{\pi}{2}$

$\Rightarrow$ Magnitude $\lim_{\omega\to\infty} |G_o(j\omega)| = \begin{cases} 0 & \text{if } r > 0 \\ |K| & \text{if } r = 0 \end{cases}$

Limit for $\omega \to -\infty$

$\Rightarrow$ Symmetric to real axis compared to $\omega \to \infty$

Nyquist Plot: Example

plot on next slide

Example

$G_o(s) = \frac{1}{s(s+1)}$

Imaginary axis $s = j\omega$

- $\omega \to 0$: $G_o(j\omega) \approx \frac{1}{j\omega}$

  $\Rightarrow \angle G_o(s) \approx -\frac{\pi}{2}$

  $|G_o(s)| \to \infty$

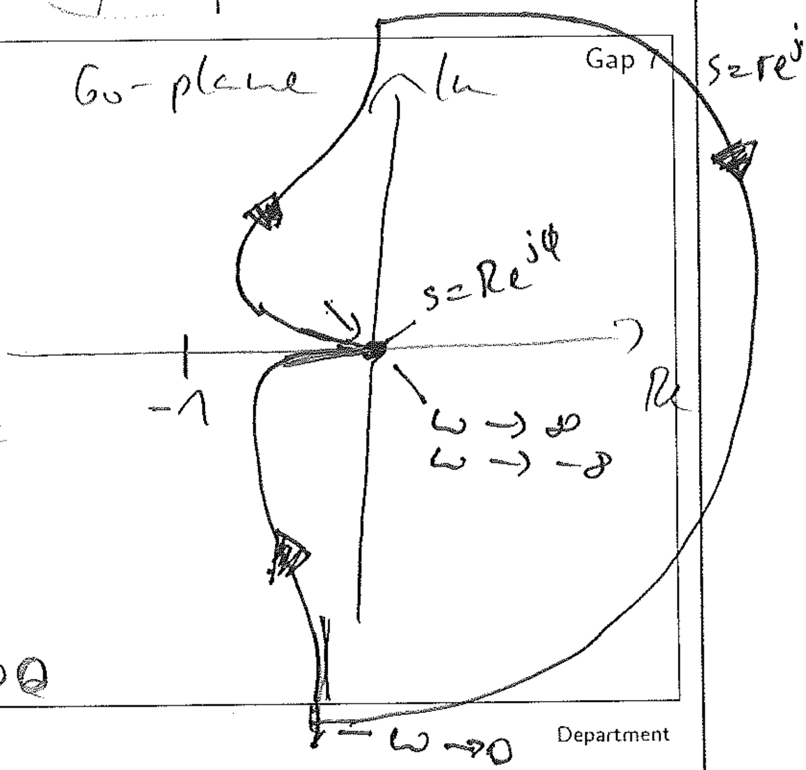
- $\omega \to \infty$: $G_o(j\omega) \approx \frac{1}{(j\omega)^2}$

  $\Rightarrow \angle G_o(s) \approx -\pi$

  $|G_o(s)| \to 0$

- $\omega \to -\infty$

  $\angle G_o(s) \approx \pi$, $|G_o(s)| \to 0$



Nyquist Plot: Example

Example $G_0(s) = \frac{(s+2)(-1)}{(s+4)(s^2+2s+2)}$

Gap 8

$\Rightarrow$ two poles on imaginary axis

$\Rightarrow$ $q = 0$ and $r = 3 - 1 = 2$

$\Rightarrow$ large semi-circle, $R = e^{j\varphi}$ maps to 0

$\Rightarrow$ no small semi-circle ($q = 0$)

$\Rightarrow$ $\omega \to 0$; $\measuredangle' G_0(s) = \pi$, $|G_0(s)| = \left|\frac{-2}{4 \cdot 2}\right| = \frac{1}{4}$

$\omega \to \infty$, $\measuredangle G_0(s) = \measuredangle K_r - r \cdot \frac{\pi}{2} = \pi - \pi = 0$

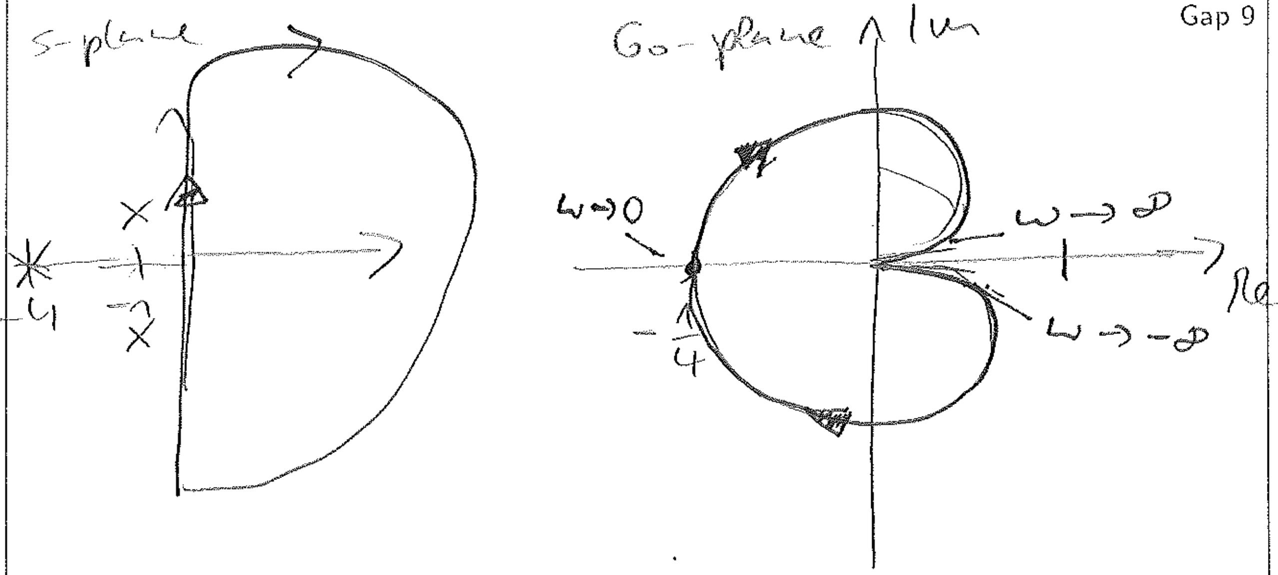
$|G_0(s)| = 0$; $G_0(j\omega) \approx -\frac{1}{(j\omega)^2} =$

$\Rightarrow$ find $s = j \Rightarrow G_0(j) = \frac{(2+j)(-1)}{(4+j)(-1+2j+2)} = -0.15 + 0.18j$

Nyquist Plot: Example

Example (Plot)

Gap 9



back

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Nyquist Plot: Example

Example $G_0(s) = \frac{1}{s^2+4} = \frac{1}{(s+2j)(s-2j)}$

Gap 10

⇒ two poles on imaginary axis: $p_{1/2} = \pm 2j$

large semi-circle: $|G_0(s)| = 0$

small semi-circle: $+2j$: $G_0(s) = \frac{1}{4j\,e^{j\varphi}}$

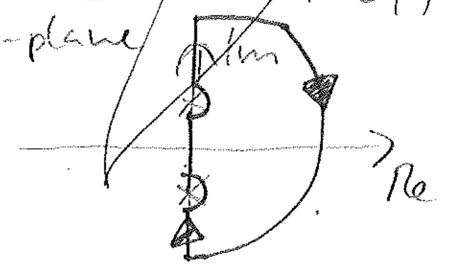
⇒ start: ∡ ... $+ \pi/2 = 0$; end: ∡ ...

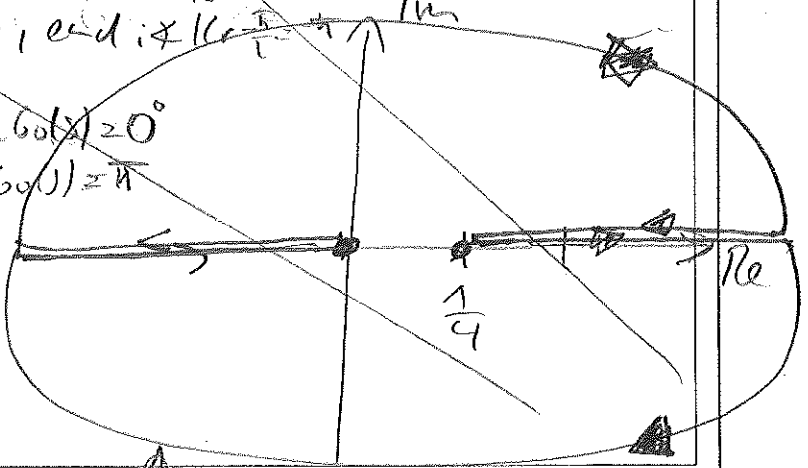
$-2j$: symmetric

$\omega \to 0$: $|G_0(s)| = 1/4$; ∡$G_0(s) = 0°$

$\omega \to \pm\infty$: $|G_0(s)| = 0$; ∡$(G_0(s)) = \pi$

s-plane





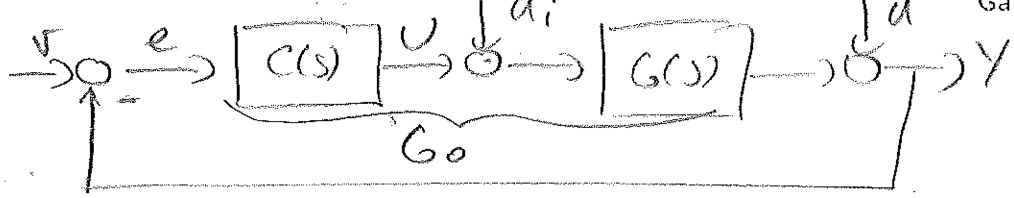
Note: $G_0(j\omega) = \frac{1}{4-\omega^2}$ on pos. real axis for $|\omega| < 1/4$, otherwise neg. R

Nyquist Plot: Stability

Nyquist Criterion

- Consider the basic feedback loop with open-loop transfer function $G_o(s) = C(s)G(s)$

Gap 11



- Call P the number of poles of G_o in the open right-half plane
- Call N the number of times the Nyquist plot of G_o encircles the point $(-1, 0)$ in clockwise direction
  ⇒ The closed loop is stable if and only if $N + P = 0$

Nyquist Plot: Stability Analysis Example

Example

Gap 12

$G_0(s) = \frac{1}{s(1+s)}$: no pole in ORHP $\Rightarrow P=0$

Nyquist plot <u>does not</u> encircle $(-1,0) \Rightarrow N=0$

$\Rightarrow P+N=0 \Rightarrow$ internally stable

$G_0(s) = -\frac{s+2}{(s+4)(s^2+2s+1)}$: $P=0$

Nyquist plot <u>does not</u> encircle $(-1,0) \Rightarrow N=0$

$\Rightarrow P+N=0 \Rightarrow$ internally stable

$G_0(s) = \frac{10s+3}{s^2+2s+4} \Rightarrow P=0$

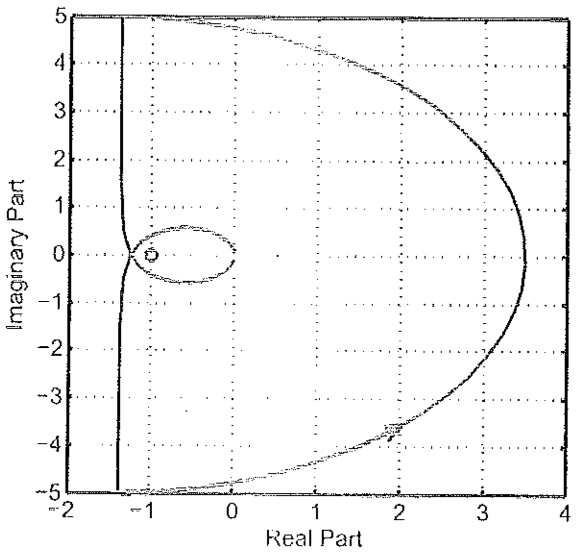
Nyquist plot does not encircle $(-1,0)$; $N=0$

$\Rightarrow P+N=0 \Rightarrow$ internally stable

Nyquist Plot: Stability Analysis Example

Instable Open Loop

- $G_o(s) = \frac{5s+2}{s(s-4)}$



Computation

Gap 13

$P = 1$ (pole at $s=4$)

$N = -1$ (one encirclement in counterclockwise direction)

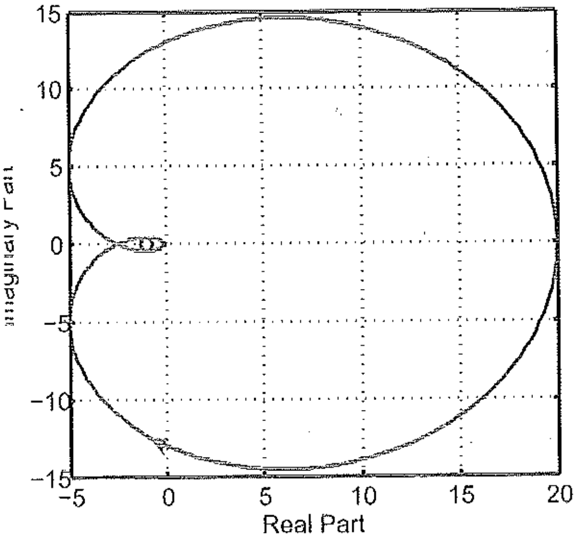
$\Rightarrow P+N = 1-1 = 0$

$\Rightarrow$ closed loop is internally stable

Nyquist Plot: Stability Analysis Example

Stable Open Loop

- $G_o(s) = \dfrac{20}{(1+s)^3}$



Computation

Gap 14

P = 0 (only stable poles)

N = 1 (one encirclement in clockwise direction)

$\Rightarrow P + N = 1 \neq 0$

$\Rightarrow$ closed loop is instable

Nyquist Plot: Gain Margin

Assumption

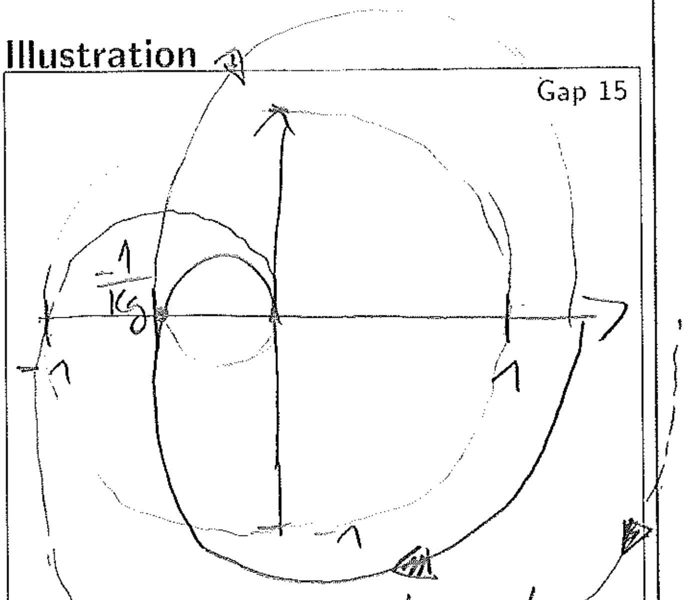
- Open loop transfer function $G_o(s)$ without poles in the open right half plane

Gain Margin

- Multiplication of G_o with constant K_G leads to instable closed loop
- Phase crossover frequency ω_P such that $\angle G_o(\omega_P) = -\pi$
  $\rightarrow$ Gain margin K_G describes degree of stability with respect to gain changes

Illustration

Gap 15



$\Rightarrow$ Multiplication by K_G leads to instability

Nyquist Plot: Phase Margin

Assumption

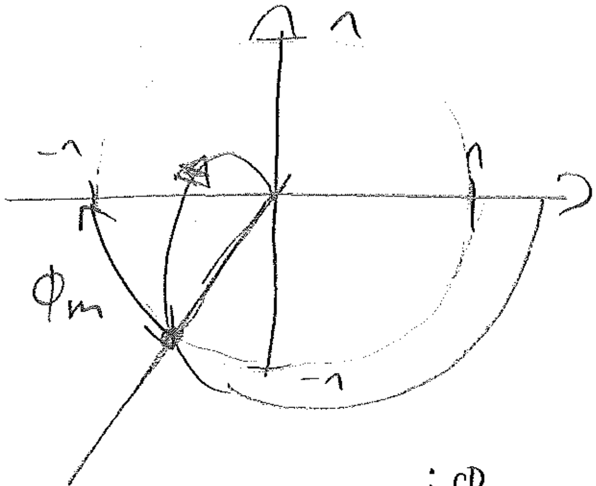
- Open loop transfer function $G_o(s)$ without poles in the open right half plane

Phase Margin

- Multiplication of G_o with $e^{-j\Phi_P}$ (phase shift of Φ_P) leads to instable closed loop
- Gain crossover frequency ω_G such that $|G_o(\omega_G)| = 1$
  $\rightarrow$ Phase margin Φ_P describes degree of stability with respect to phase shift

Illustration

Gap 16

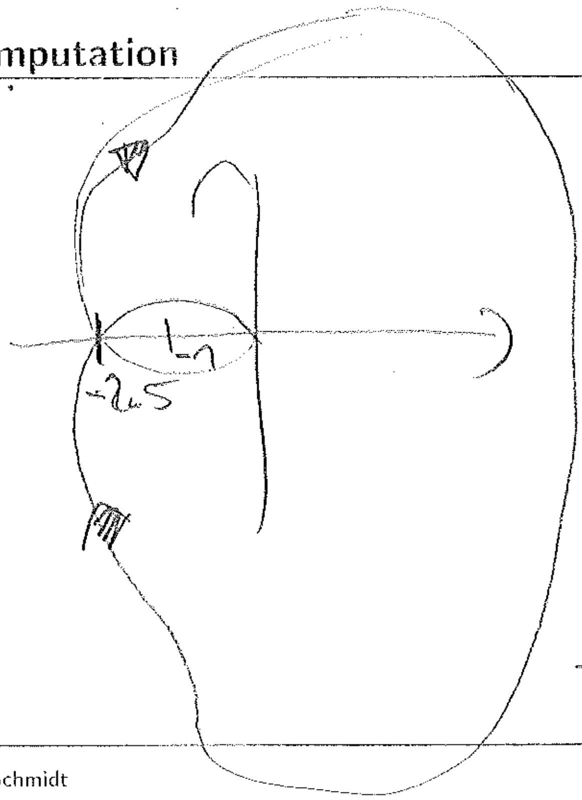


Phase-shift $e^{-j\Phi_m}$ leads to instability

Nyquist Plot: Example

Computation

Gap 17



$P = 0$, $N = 2$
$\Rightarrow$ instable
Possible Action:
Modifying G_o by proportional Gain $K_P = \frac{1}{5}$
$\Rightarrow$ Nyquist plot intersects real axis at $-1/2$
$\Rightarrow$ internally stable

$G_0(s) = \frac{10s+3}{s^2+4s+4}$

large semi-circle : r = 1 => $\lim_{R\to\infty} G_0(Re^{j\phi}) = 0$

no small semi-circle

jw-axis:

$\omega \to 0$: $|G_0(0)| = \frac{3}{4}$ $\quad \measuredangle G_0(0) = 0$

$\omega \to \infty$: $|G_0(\infty)| = 0$ $\quad \measuredangle(G_0(\infty)) = -\frac{\pi}{2}$

Intersection with real axis:

$\mathrm{Im}(G_0(j\omega)) = \mathrm{Im}\left(\frac{10j\omega+3}{-\omega^2+4j\omega+4}\right) = \mathrm{Im}\left(\frac{(10j\omega+3)(4-\omega^2-4j\omega)}{(4-\omega^2)^2+16}\right)$

$= 0 \Rightarrow 10\omega(4-\omega^2)+3(-4\omega) = \omega(-10\omega^2+40-12) =$

$= \omega(28-10\omega^2) = 0 \Rightarrow \omega^2 = \frac{14}{5}$

$\Rightarrow \mathrm{Re}(G_0(j\omega)) = 2.5$