ECE 488 – Automatic Control

Root Locus Plot and Root Locus Design

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Compulsory Course in Electronic and Communication Engineering
Credits (3/0/3)

Course Webpage: http://ECE488.cankaya.edu.tr

Reminder

Previous Topics

- Linear system modeling
- Nonlinear models and set-point linearization
- Analysis of LTI systems
- Stability and transient response
- Feedback loop

This Week

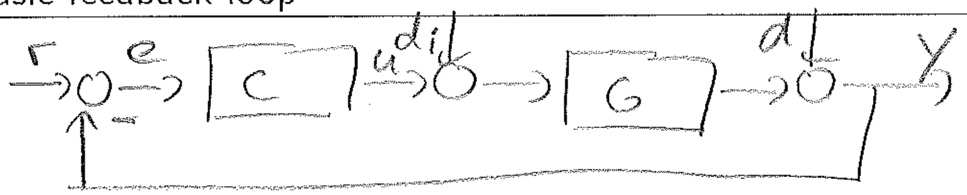
- Root locus plot
- Root locus design

Motivation: Task

Reminder

- Basic feedback loop

Gap 1



- Closed loop poles are zeros of $1 + C(s)\,G(s)$

Goal

- Determine how the poles (roots) of the feedback loop change depending on $C(s)$
- Assume that $C(s)$ is given as $C(s) = K\,C'(s)$ with a free gain parameter K

Gap 2

Motivation: Example

Computation

Gap 3

$G(s) = \frac{1}{(s-1)(s+2)}$; $C(s) = K$

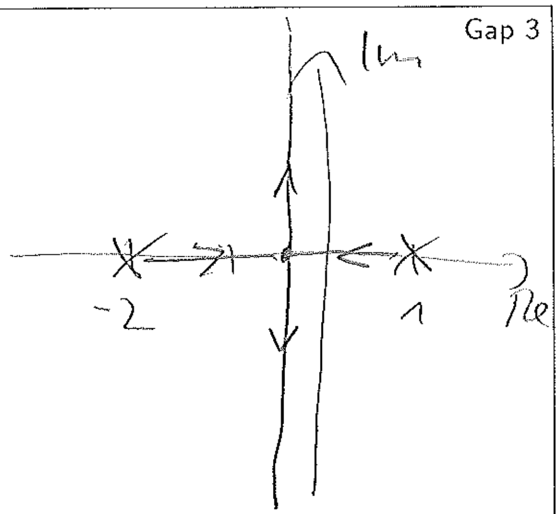
$\Rightarrow 1 + G_o(s) = 1 + \frac{K}{(s-1)(s+2)} =$

$= \frac{s^2 + s + K - 2}{(s-1)(s+2)}$

$1 + G_o(s) = 0 \Rightarrow$

$s_{1/2} = \frac{1}{2}\left(-1 \pm \sqrt{1 + 8 - 4K}\right) =$

$= -\frac{1}{2} + \frac{1}{2}\begin{cases} \sqrt{9-4K} & \text{for } K \leq \frac{9}{4} \\ j\sqrt{4K-9} & \text{for } K > \frac{9}{4} \end{cases}$



$\Rightarrow$ stable for $K > 2$

$\Rightarrow$ oscillating for $K > \frac{9}{4}$

Root Locus Construction Rules: Notation

Open Loop Transfer Function

$$G_o(s) = K\,C'(s)\,G(s) = K\,\frac{N(s)}{D(s)}$$

Pole-Zero Representation of $G_o(s)$

$$G_o(s) = K\,\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

- m zeros: $z_1, \dots, z_m$
- n poles: $p_1, \dots, p_n$
- Gain parameter $K > 0$

Root Locus Construction Rules

Root Locus Construction Rules: R1 to R3

R1

- The root locus has $\max(n, m)$ branches

R2

- The root locus starts ($K = 0$) at the poles of $G_o(s)$; there are n zeros of $D(s)$ and $m - n$ poles at $|s| = \infty$ if $m - n > 0$.
- The root locus ends ($K \to \infty$) at the zeros of $G_o(s)$; there are m zeros of $N(s)$ and $n - m$ zeros at $|s| = \infty$ if $n - m > 0$.

R3

- The root locus stays on the real axis on the left of an odd number of poles and zeros of $G_o(s)$

Root Locus Construction Rules: R4 to R5

R4

- The root locus has $n-m$ asymptotes for $|s| \to \infty$ if $n-m>0$:
  - Intersection of the asymptotes with the real axis at
    $\sigma = \dfrac{(p_1 + \dots + p_n) - (z_1 + \dots + z_m)}{n-m}$
  - Direction of the asymptotes at angles: $\theta = \dfrac{\pi}{n-m}(2\,k+1),\ k \in \mathbb{Z}$

R5

- The root locus breaks away from the real axis/joins the real axis for values of s that fulfill

$$N(s)\frac{d}{ds}D(s) - D(s)\frac{d}{ds}N(s) = 0$$

Root Locus Construction Rules: R6 to R8

R6

- The angle of departure from any complex pole p_j is

$$\pi + \angle(p_j - z_1) + \dots + \angle(p_j - z_m) - \angle(p_j - p_1) - \dots - \angle(p_j - p_n)$$

R7

- The angle of departure from any complex zero z_j is

$$\pi - \angle(z_j - z_1) - \dots - \angle(z_j - z_m) + \angle(z_j - p_1) + \dots + \angle(z_j - p_n)$$

R8

- For intersections of the root locus with the imaginary axis, it holds that $D(s) + K\,N(s)$ is divided by $s^2+\omega^2$ for some K and ω

Root Locus Construction Rules: General Remarks

Application of the Rules

- In general, not all rules need to be used (some rules might not be applicable)
- The root locus plot gives information about the poles of the closed loop
- The root locus plot is constructed using the open-loop transfer function $G_o(s)$
- For a root s^* on the root locus, the corresponding value K is computed as

$$K = -\frac{D(s^*)}{N(s^*)}$$

- The construction of the root locus plot is formulated for the gain K as free parameter but any free parameter could be used

Explanation: R1 and R2

R1

Gap 4

$1 + G_o(s) = 0 \Leftrightarrow K \cdot N(s) + D(s) = 0$

$\Rightarrow$ if $m \geq n$: m zeros

$\Rightarrow$ if $n > m$: n zeros

$\max(n, m)$

R2

Gap 5

$1 + G_o(s) = 0 \Leftrightarrow K + \frac{D(s)}{N(s)} = 0$

For $K = 0$: $D(s) = 0$ or $\frac{D(s)}{N(s)} \to 0$ for $|s| \to \infty$ if $m > n$

$1 + G_o(s) = 0 \Leftrightarrow \frac{1}{K} + \frac{N(s)}{D(s)} = 0$

For $K \to \infty$: $N(s) = 0$ or $\frac{N(s)}{D(s)} \to 0$ for $|s| \to \infty$ if $n > m$

Explanation: R3

add one slide for plot

R3

Gap 6

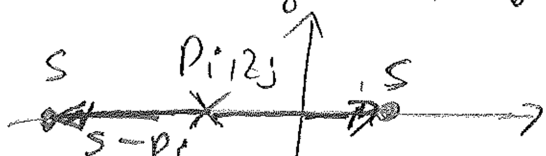
We want $\frac{N(s)}{D(s)} = \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} = -\frac{1}{K}$

Phase condition:

$\measuredangle(s-z_1)+\cdots+\measuredangle(s-z_m) - \measuredangle(s-p_1) - \cdots$

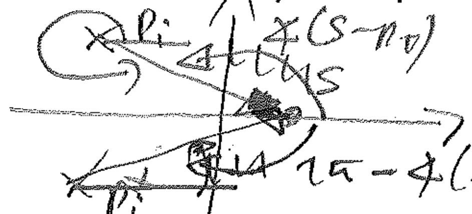
$-\measuredangle(s-p_n) = (2h+1)\cdot\pi, \; h\in\mathbb{Z}$

s on the right/left of p_i / z_j



$\measuredangle(s-p_i) = \pi \qquad \measuredangle(s-p_i) = 0$

s and complex pair



$\measuredangle(s-p_i) + \measuredangle(s-p_i^*) = 2\pi$

Explanation: R4

R4

Gap 7

Approximation for $|s| \to \infty$:

$G_0(s) = K\frac{N(s)}{D(s)} \approx K\cdot\frac{s^m}{s^n} = -1$

$\Rightarrow \measuredangle\left(\frac{s^m}{s^n}\right) = \measuredangle(s^{m-n}) = \pi(2h+1), \; h\in\mathbb{Z}$

$\Rightarrow$ with $s = Me^{-j\theta}$

$\Rightarrow \measuredangle(s^{n-m}) = \measuredangle\left(e^{j(n-m)\theta}\right) = (n-m)\theta = (2h+1)\pi$

$\Rightarrow \theta = \frac{2(h+1)\pi}{n-m}$ (n−m asymptotes)

$\leadsto$ computation of σ more involved

Explanation: R5 and R6

R5

Gap 8

→ we need a double zero of $1+G_o(s)$

⇒ $\frac{d}{ds}\left(1+K\frac{N(s)}{D(s)}\right) = K\frac{D(s)\frac{d}{ds}N(s) - N(s)\frac{d}{ds}D(s)}{D(s)^2} = 0$

⇒ $D(s)\frac{d}{ds}N(s) - N(s)\frac{d}{ds}D(s) = 0$

R6

Gap 9

We want $\lim_{s\to p_j}(s-p_j)$. From the angle condition, we get

$$\lim_{s\to p_j} \angle(s-p_j) = \angle(p_j-z_1)+\dots+\angle(p_j-z_m) - \angle(p_j-p_1) - \dots - \angle(p_j-p_{n}) + \pi$$

⇝ same derivation for zero R7

Examples: $G_o(s) = K\frac{s+3}{s(s^2+2s+5)}$

→ add one sketch for plot

Computation

Gap 10

R1: max(3,1) = 3 branches

R2: start at $s=0$, $s=-1\pm 2j$

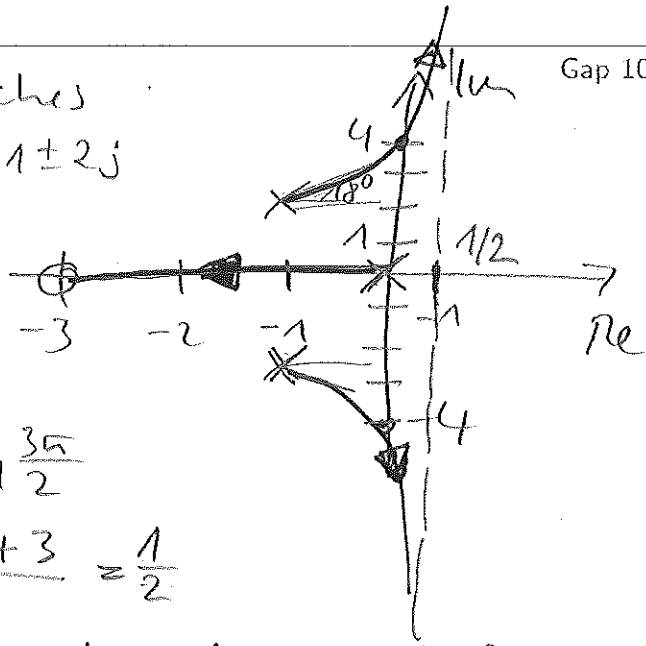
end at $s=-3$

and $|s|\to\infty$ (2 branches)

R3: → see plot

R4: $\Theta = \frac{(2h+1)\pi}{2} = \frac{\pi}{2}, \frac{3\pi}{2}$

$\sigma = \frac{0-1-\cancel{2j}-1+\cancel{2j}+3}{2} = \frac{1}{2}$



R5: does not apply; no branches can join on real axis

Examples: $G_o(s) = K\,\frac{s+3}{s\,(s^2+2\,s+5)}$

Computation

Gap 11

R6: angle at pole $p_1 = -1+2j$

$$\lim_{s \to p_1} \measuredangle(s-p_1) = 180^\circ + \measuredangle(-1+2j+3) - \measuredangle(-1+2j-0) - \measuredangle(-1+2j+1+2j) = 180^\circ + 45^\circ - 117^\circ - 90^\circ = 18^\circ$$

R8: We divide $D(s) + K\,N(s)$ by $s^2+\omega^2$

$$(s^3 + 2s^2 + (5+K)s + 3K) / (s^2+\omega^2) = s+2$$

s^3, $2s^2$, $\omega^2 s$, $2\omega^2$ $\Rightarrow$ $5+K = \omega^2$ and $3K = 2\omega^2$

$\Rightarrow 3K = 10 + 2K \Rightarrow K = 10$ and $\omega = \sqrt{15} \approx 4$

$\Rightarrow$ intersection with imag axis at $s = \pm j4$

Examples: $G_o(s) = K\,\frac{s+1}{s^2\,(s+9)}$

add one slide for plot?

Computation

Gap 12

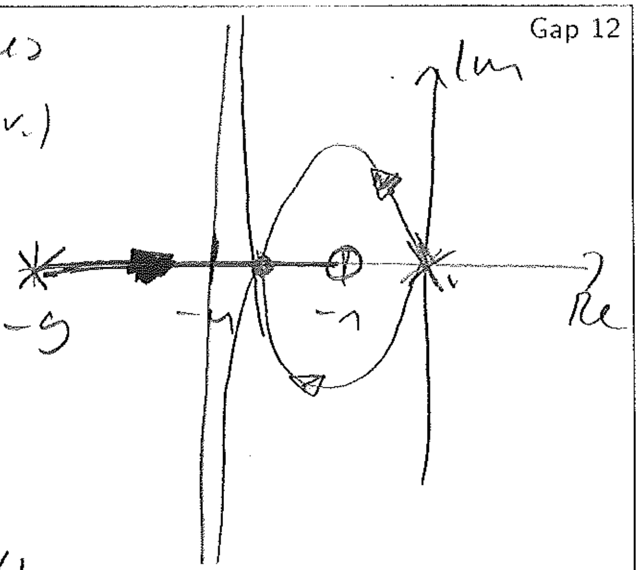
R1: $\max(3,1) = 3$ branches

R2: start at $s=0$ (2 br.) and $s=-9$, end at -1 and $|s| \to \infty$ (2 br.)

R3: → see plot

R4: $\Theta = \frac{(2k+1)\pi}{2}$

$\sigma = \frac{-9+0+0+1}{2} = -4$

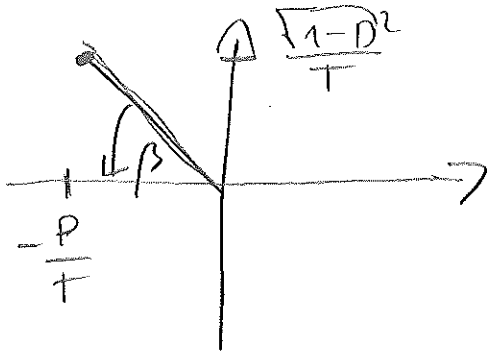


Root Locus Design: Dominant Complex Poles

Illustration

Gap 17

closed-loop pole location: $p = -\frac{D}{T} \pm j\frac{\sqrt{1-D^2}}{T}$



$\beta = \arctan \frac{\sqrt{1-D^2}}{D}$

$\Rightarrow$ all poles with damping D lie on straight line

$\Rightarrow$ if D is chosen, oscillation frequency $\frac{\sqrt{1-D^2}}{T}$ and exp. decay $-\frac{D}{T}$ only depend on T

Design Problem

- Shape the root locus plot to achieve the desired pole locations

Root Locus Design: Assignment of K

Procedure

- Sketch requirements in root locus plot
- Choose appropriate root locus $\rightarrow$ compute K

Example

Gap 18

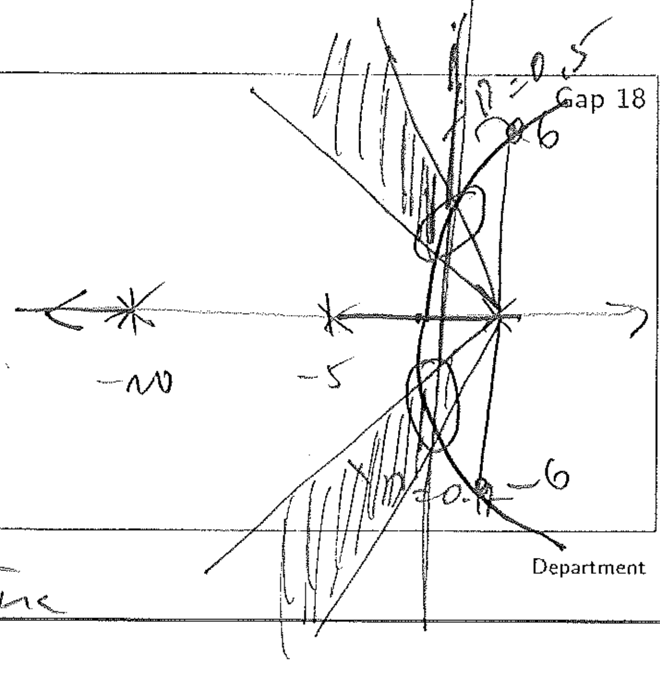
$G_0(s) = \frac{4K}{s(s+10)(s+5)}$

Specifications

$0.7 \geq D \geq 0.5$

Settling time $t_s = 4$

$\Rightarrow -\frac{D}{T} = -\frac{4}{4} = -1$



$\Rightarrow$ choose pole in shaded area

Root Locus Design: Description

Given Open Loop Transfer Function $G_o(s)$

- Construct root locus plot of $G_o(s)$
- Choose *desirable* closed-loop pole locations
- Compute K from desirable pole locations

Given Plant Transfer Function $G(s)$

- Construct root locus plot of $G(s)$
- Choose *desirable* closed-loop pole locations
- Choose controller poles/zeros such that root locus plot fulfills desirable properties
- Compute K from desirable pole locations

Root Locus Design: Dominant Complex Poles

Performance Specification

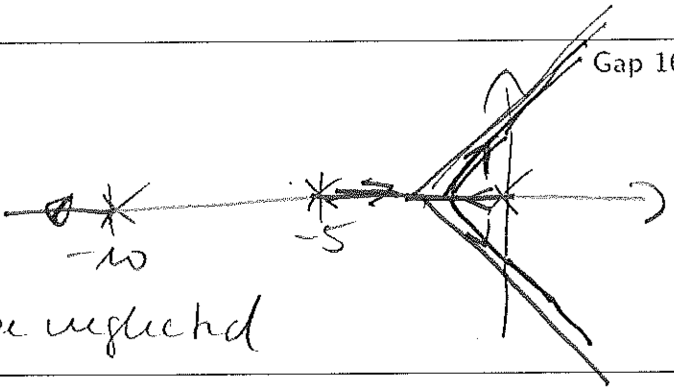
- Rise time $t_r = \frac{T}{\sqrt{1-D^2}}(\pi - \arctan(\frac{\sqrt{1-D^2}}{D}))$
- Damping D
- Settling time $t_s = \frac{3\,T}{D}$ (5%) or $t_s = \frac{4\,T}{D}$ (2%)
  $\Rightarrow$ Determine desired closed-loop pole locations

Illustration

Gap 16

Example

$G_0(s) = \frac{416}{s(s+10)(s+5)}$



$\rightsquigarrow$ pole at -10 can be neglected

Root Locus Design: Controller Choice

Procedure

- Sketch the root locus of the plant transfer function and check if the desired properties (damping, overshoot, oscillations, decay) can be fulfilled
- Add controller poles/zeros to change the root locus plot in order to fulfill the desired properties

Example

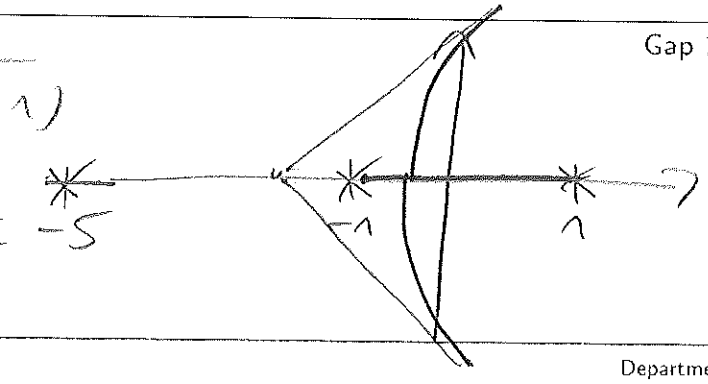
Gap 19

$G_0(s) = \frac{1}{(s+5)(s-1)(s+1)}$

R4: $\theta = \frac{(2h+1)\pi}{3}$

$\sigma = \frac{-5+1-1}{3} = -\frac{5}{3}$

R5: $s_{1/2} \approx -0.1$



Root Locus Design: Example

plot on next slide ?

Computation

Gap 20

Specification: no static position error, $t_s \leq 8$

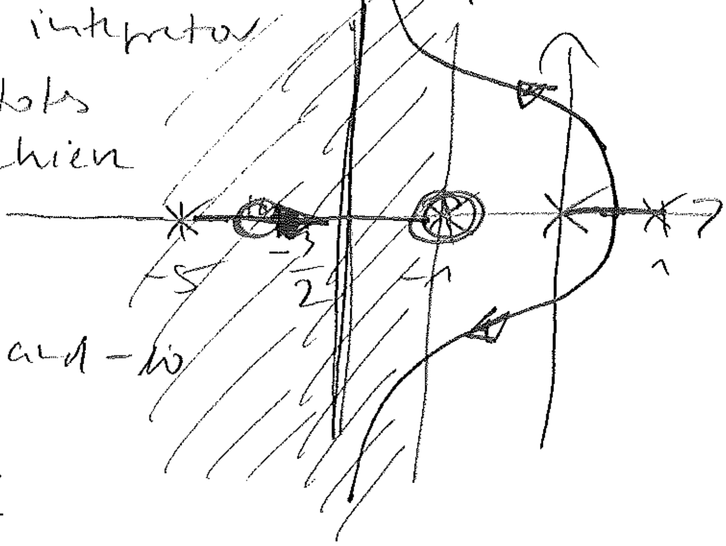
$\Rightarrow -\frac{p}{r} \leq -1$ and integrator

Thus: move asymptotes to the left and achieve angle $\pi/2$

$\Rightarrow$ put two zeros, for example at -1 and -10

$\Rightarrow \theta = \pi/2$

$\sigma = \frac{-5+1+1}{2} = -\frac{3}{2}$



Examples: $G_o(s) = K \frac{s+1}{s^2\,(s+9)}$

Computation

Gap 13

R5: $D(s)\frac{dN}{ds} - N(s)\frac{dD(s)}{ds} = (s^3+9s^2)\cdot 1 - (s+1)(3s^2+18s) =$

$= s(s^2+9s-3s^2-21s-18) = s(-2s^2-12s-18) = 0$

$\Rightarrow s=0$ and $s=-3$

R6 and R7 do not apply

R8 only intersection with imaginary axis at $s=0$

Arbitrary Free Parameter: Description

maybe remove

Open Loop Transfer Function

$$G_o(s) = K\,\frac{N(s)}{D(s)} = K\,\frac{n_m\,s^m + \cdots + n_0}{d_n\,s^n + \cdots + d_0}$$

Free Parameter

- K: Classical root locus construction
- Other parameter: Reformulation of G_o

Example: d_i **as free parameter**

$$1 + G_o(s) = 0 \Rightarrow K\,N(s) + d_i\,s^i + (D(s) - d_i\,s^i) = 0$$

$$\Rightarrow 1 + \frac{d_i\,s^i}{K\,N(s) + D(s) - d_i\,s^i} = 1 + G_o'(s) = 0$$

$\Rightarrow$ Perform root locus construction with $G_o'(s)$ and d_i as free parameter

Arbitrary Free Parameter: $G_o(s) = \frac{s+1}{s\,(s+d)}$

Computation

Gap 14

$\Rightarrow 1 + \frac{s+1}{s(s+d)} = 0$

$\Rightarrow s^2 + sd + s + 1 = 0$

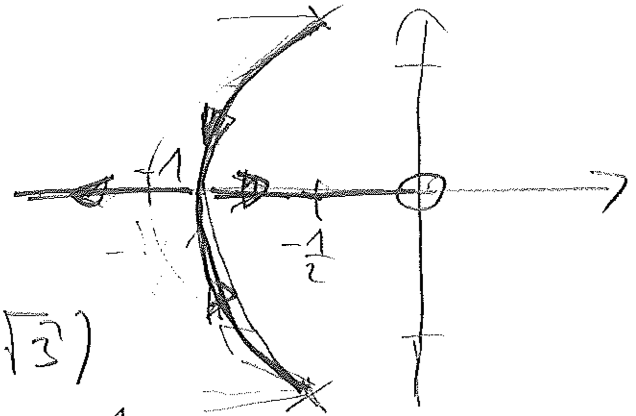
$\Rightarrow 1 + \frac{sd}{s^2+s+1} = 0$

$\Rightarrow G_o'(s) = \frac{s}{s^2+s+1}$

poles: $s_{1/2} = \frac{1}{2}(-1 \pm j\sqrt{3})$

R1: $\max(2,1) = 2$ branches

R2: start at $s_{1/2} = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$



Arbitrary Free Parameter: $G_o(s) = \frac{s+1}{s\,(s+d)}$

Computation

Gap 15

end at $s = 0$, $|s| \to \infty$

R3 → see plot, negative real axis

R4: $\Theta = \frac{(2h+1)\pi}{1} = \pi$

R5: $(s^2+s+1)\cdot 1 - s(2s+1) = -s^2+1 = 0$
$\Rightarrow s = \pm 1$

R6: $\lim_{s\to p_1} \angle(s-p_1) = 180^\circ - \angle\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2} + \frac{1}{2} + j\frac{\sqrt{3}}{2}\right) +$
$+ \angle\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) = 180^\circ - 90^\circ + 120^\circ = 210^\circ$

R8: no intersection