ECE 488 – Automatic Control

Steady-State and Transient Response of LTI Systems

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Compulsory Course in Electronic and Communication Engineering
Credits (3/0/3)

Course Webpage: http://ECE488.cankaya.edu.tr

Reminder

Previous Topics

- Linear system modeling
- LTI ODEs, block diagrams, transfer functions, state space models
- Block diagram simplification
- Nonlinear models and set-point linearization
- Analysis of Transfer Functions and Stability

This Week

- Steady-state response and Transient Response
- Dominant Poles
- Transfer Function Zeros
- Performance Specifications

Responses: Separation

Transient Response

Response of a system to an input signal for a short time period after the application of the input signal: $y_{tr}(t)$

Steady-State Response

Long term response of a system to an input signal after the transient response vanishes: $y_{ss}(t)$

Separation

$$y(t) = y_{tr}(t) + y_{ss}(t)$$

Remark

If stability of a control system is ensured, it is desired to shape the transient response of the control system

Responses: Example

Computation

Gap 1

$$G(s) = \frac{A}{s+a} + \frac{B}{s+b}; \quad U(s) = \frac{1}{s} \quad (\text{step})$$

$$\Rightarrow Y(s) = \frac{A}{(s+a)s} + \frac{B}{(s+b)s} = \frac{A/a}{s} - \frac{A/a}{s+a} + \frac{B/b}{s} - \frac{B/b}{s+b}$$

$$\Rightarrow y(t) = \sigma(t)\left(\frac{A}{a} + \frac{B}{b} - \frac{A}{a}e^{-at} - \frac{B}{b}e^{-bt}\right) =$$

$$= \underbrace{\sigma(t)\left(\frac{A}{a} + \frac{B}{b}\right)}_{y_{ss}(t)} + \underbrace{\sigma(t)\left(-\frac{A}{a}e^{-at} - \frac{B}{b}e^{-bt}\right)}_{y_{tr}(t)}$$

$$\Rightarrow \lim_{t \to \infty} y_{tr}(t) = 0 \quad \text{for } a, b > 0$$

Responses: Properties

Properties

- If $G(s)$ is BIBO stable, then the transient response converges to zero

$$\lim_{t\to\infty} y_{tr}(t) = 0$$

- If $G(s)$ is instable, then the transient response diverges

$$\lim_{t\to\infty} |y_{tr}(t)| = \infty$$

- Steady state response for LTI systems is determined by the input

Example

Gap 2

$$G(s) = \frac{1}{s+1}, \quad U(s) = \frac{2}{s^2+4} \quad (u(t) = \sin 2t)$$

$$\Rightarrow Y(s) = \frac{2}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+4}$$

Responses: Example

Computation

Gap 3

some computation, $A = \frac{2}{5}$, $B = -\frac{2}{5}$, $C = 2/5$

$$\Rightarrow Y(s) = \frac{1}{5}\left(\frac{2}{s+1} + \frac{2}{s^2+4} - \frac{2s}{s^2+4}\right)$$

$$\Rightarrow y(t) = \frac{1}{5}\sigma(t)\left(2e^{-t} + \sin 2t - 2\cos 2t\right)$$

$$= \frac{1}{5}\sigma(t)\left(\underbrace{2e^{-t}}_{y_{tr}} + \underbrace{\sqrt{5}\sin(2t + \arctan(-2))}_{y_{ss}}\right)$$

Properties of the Transient Response: Distinct Poles

Step Response

$$Y(s) = G(s)\,\frac{1}{s} = \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)}\,\frac{1}{s}$$

Decomposition for Distinct Poles

Gap 4

Residuum r_i for pole p_i:

$$r_i = \lim_{s \to p_i} Y(s)\,(s-p_i) =$$

$$= \lim_{s \to p_i} \frac{G(s)}{s}(s-p_i)$$

$$\Rightarrow Y(s) = \frac{r_0}{s} + \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n}$$

$\rightarrow$ $p_1, p_2, \ldots, p_n$ are called the modes of the system

Properties of the Transient Response: Example

Computation

Gap 5

$$Y(s) = \frac{s+c}{(s+4)\cdot s} \quad ; \; p_1 = -4 \,|\, p_0 = 0$$

$$\Rightarrow r_0 = \lim_{s \to 0} \frac{(s+c)\cdot s}{(s+4)\cdot s} = \frac{c}{4}$$

$$r_1 = \lim_{s \to -4} \frac{(s+c)(s+4)}{(s+4)\cdot s} = \frac{-4+c}{-4} = 1 - \frac{c}{4}$$

$$\Rightarrow Y(s) = \frac{1 - c/4}{s+4} + \frac{c/4}{s}$$

Properties of the Transient Response: Cases

Real Pole $p_i \neq 0$

$$\frac{r_i}{s - p_i} \bullet\!\!-\!\!\circ\; r_i\, e^{p_i t}\sigma(t)$$

$$\Rightarrow \lim_{t\to\infty} r_i\, e^{p_i t}\sigma(t) = \begin{cases} 0 & \text{if } p_i < 0 \\ \infty & \text{if } p_i > 0 \end{cases}$$

Comparison $p_i < p_j < 0$ **and** $a_i \approx a_j$

$$r_i\, e^{p_i t}\sigma(t) < r_j\, e^{p_j t}\sigma(t)$$

$\Rightarrow p_j$ dominates p_i

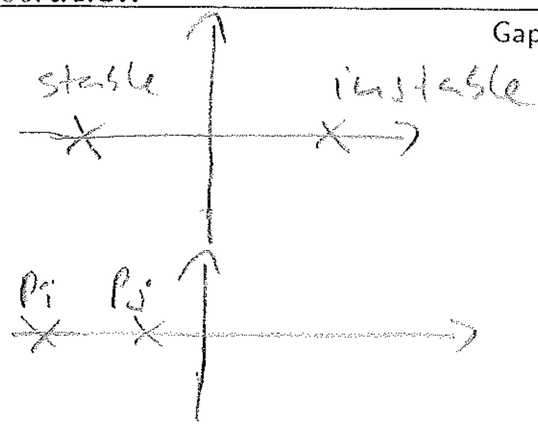
Comparison p_i, p_j **but** $r_j << r_i$

$$r_j\, e^{p_j t}\sigma(t) < r_i\, e^{p_i t}\sigma(t)$$

$\Rightarrow p_i$ dominates p_j

Illustration

Gap 6



mode p_i converges to zero faster than mode p_j

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Properties of the Transient Response: Example

Output Response

$$Y(s) = \frac{A/a + B/b}{s} - \frac{A/a}{s+a} - \frac{B/b}{s+b}$$

Computation

Gap 7

$a = 2;\ b = 10 \Rightarrow p_1 = -10 < p_2 = -2$

Assume $A = 2;\ B = 10$

$\Rightarrow y(t) = \sigma(t)\left(2 - e^{-2t} - e^{-10t}\right)$; p_2 dominates p_1

Assume $A = 2,\ B = 200$

$\Rightarrow y(t) = \sigma(t)\left(21 - e^{-2t} - 20e^{-10t}\right)$

$\rightarrow p_1$ dominates p_2

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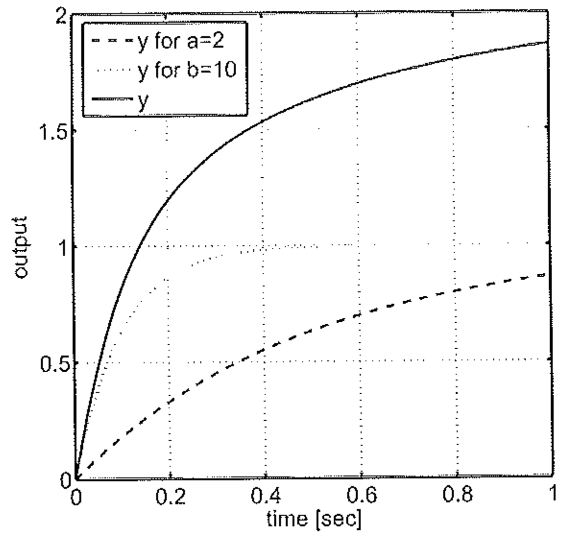
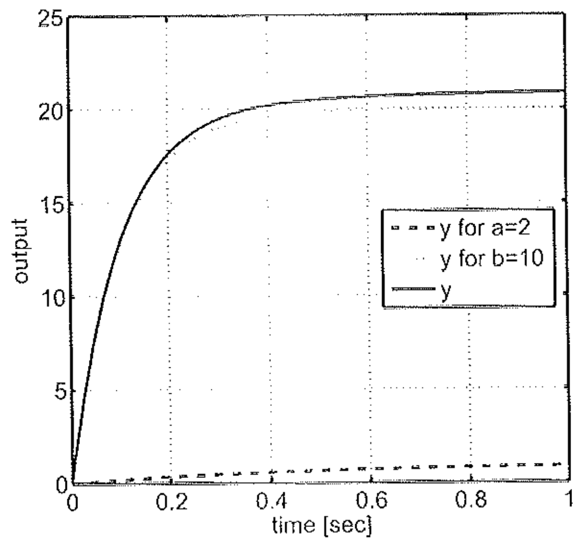
Properties of the Transient Response: Example

$A/a = B/b$ | $A/a << B/b$





$\Rightarrow$ pole at a dominates pole at b

$\Rightarrow$ pole at b dominates pole at a

Properties of the Transient Response: Cases

Conjugated Complex Poles

$$\frac{r_i}{s-p_i} + \frac{r_i^\star}{s-p_i^\star} \;\bullet\!\!-\!\!\circ\; |r_i|\, e^{\mathrm{Re}(p_i)\,t} \sin(\mathrm{Im}(p_i)\,t - \angle(r_i))$$

Computation

Gap 8

$$\frac{r_i}{s-p_i} + \frac{r_i^*}{s-p_i^*} \;\bullet\!\!-\!\!\circ\; r_i e^{+p_i t} + r_i^* e^{+p_i^* t}$$

$$= |r_i| e^{j\angle(r_i)} e^{Re(p_i)t} e^{j Im(p_i)t} + |r_i| e^{-j\angle(r_i)} e^{Re(p_i)t} e^{-j Im(p_i)t}$$

$$= |r_i| e^{Re(p_i)t} \left(e^{j(Im(p_i)t + \angle r_i)} + e^{-j(Im(p_i)t + \angle(r_i))}\right)$$

$$= 2|r_i| e^{Re(p_i)t} \cdot \cos(Im(p_i)t + \angle(r_i))$$

Properties of the Transient Response: Cases

Properties

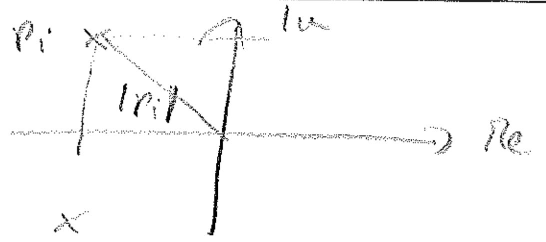
$$\lim_{t\to\infty} |r_i|\, e^{\mathrm{Re}(p_i)\,t} \sin(\mathrm{Im}(p_i)\,t - \angle(r_i)) = \begin{cases} 0 & \text{if } \mathrm{Re}(p_i) < 0 \\ \infty & \text{if } \mathrm{Re}(p_i) > 0 \end{cases}$$

$\Rightarrow$ exponential decay/increase similar to real pole

$$D = \frac{\mathrm{Re}(p_i)}{|p_i|} \text{ (damping)}$$

Computation

Gap 9



Damping decreases if move far away from real axis

Dominant Poles: Relation

Conditions for Poles

- If there is an instable pole, it dominates all stable poles
- Usually, stable poles close to the imaginary axis (slow convergence) dominate stable poles far from the imaginary axis (fast convergence)
- Exceptions exist depending on the residues of the modes

Examples

Gap 10

$G_1(s) = \frac{s+2}{(s-1)(s+10)}$ $G_2(s) = \frac{1}{(s+1)(s+10)}$

instable pole at $s=1$ dominates

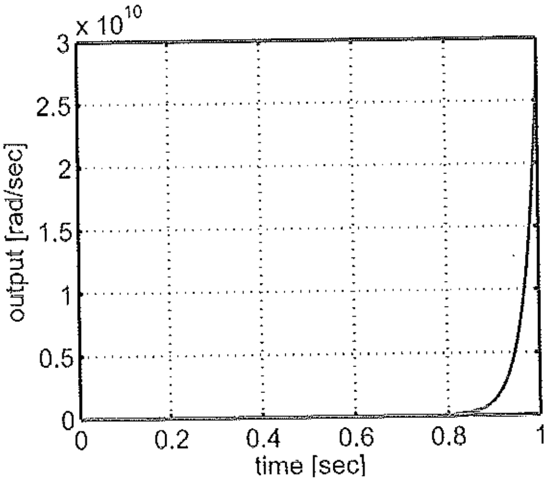
stable pole $s=-1 > -10$ dominates

Dominant Poles: Simulation

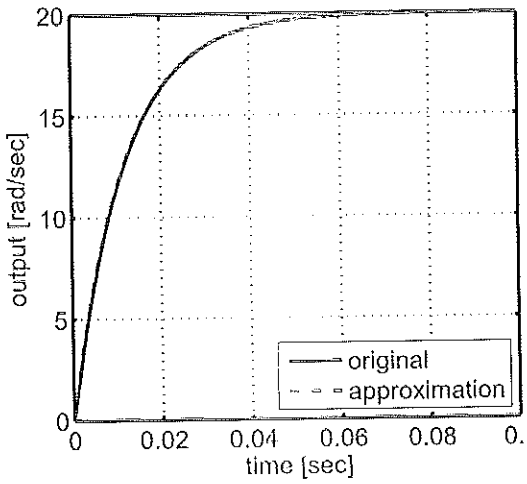
Gap 11

$$G(s) = \frac{K_i/u}{\left(s+\sqrt{\frac{K_i}{u}}\right)\left(-s+\sqrt{\frac{K_i}{u}}\right)} \qquad G(s) \approx \frac{20}{(s+85)(s+5000)}$$

Magnetic Suspension



DC Motor



Plant Zeros: Basics

Stable Plant

$$G(s) = \frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}$$

Minimum-Phase Zero

- Zeros in the open left half plane: $\mathrm{Re}(z_j) < 0$

Non-minimum Phase Zero

- Zero in the right half plane: $\mathrm{Re}(z_j) > 0$

Example

Gap 12

$G(s) = 5 \cdot \frac{1-s}{1+s} \cdot \frac{1}{1+30s}$ has one

non-minimum phase zero at $s = 1$

Plant Zeros: Minimum Phase Zeros and Dominant Poles

Suppression of Dominant Poles

- Assume a zero z_j is close to the dominant pole p_i: $z_j \approx p_i$

$$G(s) = \frac{(s - z_j)\tilde{B}(s)}{(s - p_i)\tilde{A}(s)}$$

- Residue of mode p for step response

$$r_i = \lim_{s \to p_i} \frac{G(s)\,(s - p_i)}{s} = \lim_{s \to p_i} \frac{(p_i - z_j)\tilde{B}(p_i)\,(s - p_i)}{(s - p_i)\tilde{A}(p_i)} \approx 0$$

$\Rightarrow$ Dominant mode with pole p_i does not appear in the step response
$\Rightarrow$ If $z_j \approx p_i$, dominant modes can be suppressed and other modes become dominant

Plant Zeros: Minimum Phase Zeros and Dominant Poles

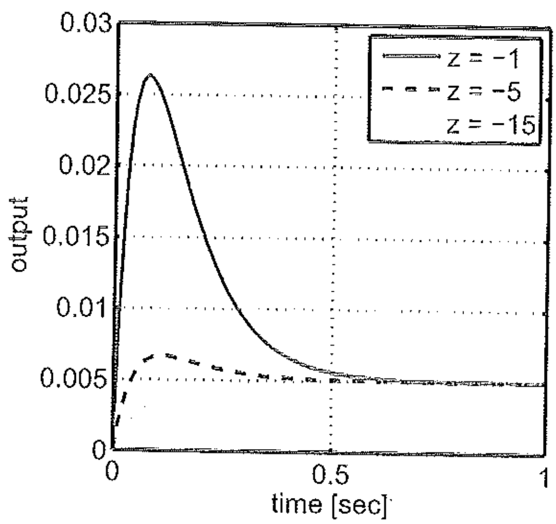
Overshoot

- Dominant plant pole at p with $\mathrm{Re}(p) < 0$
- Slow minimum phase plant zero z with $\mathrm{Re}(p) << \mathrm{Re}(z) < 0$

$\Rightarrow$ Overshoot of the step response

Example Transfer Function

$$G(s) = \frac{-1}{z}\,\frac{(s - z)}{(s + 10)(s + 20)}$$



Plant Zeros: Non-minimum Phase

Statement

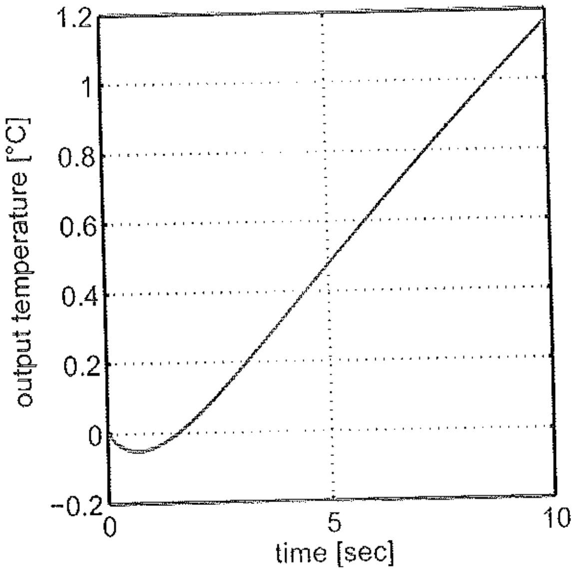
- k non-minimum phase zeros in $G(s)$
  $\Rightarrow$ Step response intersects with time-axis k times
  $\Rightarrow$ Undershoot whenever there are non-minimum phase zeros

Example

$$G(s) = \frac{5(s-1)}{(s+1)(1+30s)}$$

$\Rightarrow$ One intersection with time axis

Example Simulation

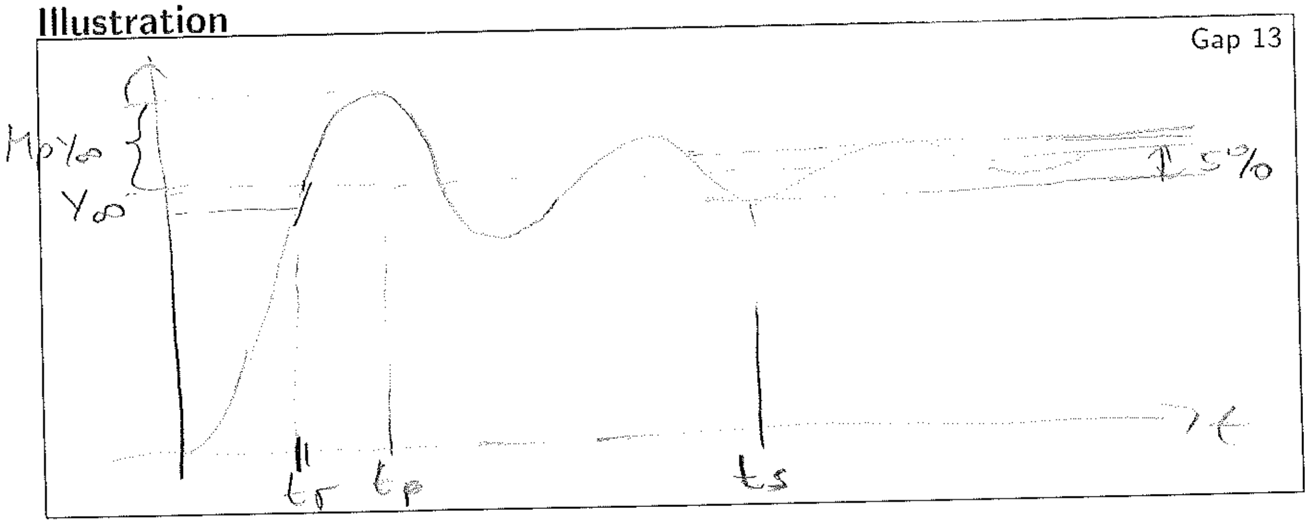


Performance Specifications: Step Response Characteristics

Performance Specifications

- Describe desired behavior of a control system based on step response
- Define performance metrics that can be used in practice

Illustration



Performance Specifications: Step Response Characteristics

Steady State Value

- Output in steady state: $y_\infty = \lim_{t\to\infty} y(t)$

Rise Time

- Quantifies speed of response: first time t_r such that $y(t_r) = 0.95 y_\infty$

(Percent) Overshoot

- Quantifies damping of response: $M_p = \max_{t\in\mathbb{R}} \frac{y(t) - y_\infty}{y_\infty}$

Peak Time

- Time until first peak of overshoot is reached: t_p

Settling Time

- Quantifies how long it takes until response stays around the final value (for example 2% or 5%): t_s

Performance Specifications: Example

First-order Lag

$$Y(s) = \frac{K}{1+sT}\frac{1}{s} \bullet\!\!-\!\!\circ\; y(t) = \sigma(t)(1 - e^{-t/T})$$

Computation

Gap 14

t_r: $K(1-e^{-t/T}) = 0.95K$

$\Rightarrow e^{-t/T} = 0.05 \Rightarrow -t/T = \ln 0.05$

$\Rightarrow t_r = -T \ln 0.05$

M_p, t_p does not exist

$t_s(5\%) = t_r$

Performance Specifications: Example

Second-order Lag ($D < 1$)

$$Y(s) = \frac{K}{1 + 2\,D\,T\,s + T^2 s^2}\,\frac{1}{s}$$

$$y(t) = \sigma(t) K \big(1 - \frac{e^{-Dt/T}}{\sqrt{1-D^2}} \sin(\frac{\sqrt{1-D^2}}{T} t + \arctan\frac{\sqrt{1-D^2}}{D})\big)$$

Computation

Gap 15

For $D > 0$, $y_\infty = K$

t_r: $1 - \frac{e^{-Dt/T}}{\sqrt{1-D^2}} \sin(\frac{\sqrt{1-D^2}}{T} t_r + \arctan\frac{\sqrt{1-D^2}}{D}) = 1 - 0.95 \approx 0$

$\Rightarrow \arctan\frac{\sqrt{1-D^2}}{D} \approx -\frac{\sqrt{1-D^2}}{T} \cdot t_r \Rightarrow t_r \approx -\frac{T}{\sqrt{1-D^2}} \arctan\frac{\sqrt{1-D^2}}{D}$

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Performance Specifications: Example

Computation

Gap 16

t_p: $\frac{dy}{dt} = 0 = K\Big(\frac{e^{-Dt/T}}{\sqrt{1-D^2}} \cdot \frac{D}{T} \sin(\frac{\sqrt{1-D^2}}{T} t + \arctan\frac{\sqrt{1-D^2}}{D})$

$- \frac{e^{-Dt/T}}{\sqrt{1-D^2}} \cdot \frac{\sqrt{1-D^2}}{T} \cos(\frac{\sqrt{1-D^2}}{T} t + \arctan\frac{\sqrt{1-D^2}}{D})\Big)$

$\Rightarrow \frac{D}{\sqrt{1-D^2}\,T} \sin(\cdot) = \frac{1}{T} \cos(\cdot)$

$\Rightarrow \tan(\frac{\sqrt{1-D^2}}{T} t + \arctan\frac{\sqrt{1-D^2}}{D}) = \frac{\sqrt{1-D^2}}{D}$

$\Rightarrow \frac{\sqrt{1-D^2}}{T} \cdot t + \arctan\frac{\sqrt{1-D^2}}{D} = \arctan\frac{\sqrt{1-D^2}}{D} + k \cdot \pi$

$\Rightarrow$ first Max for $k = 1$ $\Rightarrow t_p = \frac{\pi T}{\sqrt{1-D^2}}$

$\Rightarrow h_p = e^{-\frac{D}{\sqrt{1-D^2}} \cdot \pi}$

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Settling time, 5%: $t_s \approx \frac{3T}{D}$
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2%: $t_s \approx \frac{4T}{D}$