ECE 488 – Automatic Control

Properties of LTI Systems

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Compulsory Course in Electronic and Communication Engineering
Credits (3/0/3)

Course Webpage: http://ECE488.cankaya.edu.tr

Reminder

Previous Topics

- Linear system modeling
- LTI ODEs, block diagrams, transfer functions, state space models
- Block diagram simplification
- Nonlinear models and set-point linearization

This Week

- Solution of the state equations
- Analysis of transfer functions
- Stability

Solution of the State Equation: Basics

State Equations

$$\dot{x} = A\,x + b\,u, \quad x(0) = x_0$$
$$y = c^T x + d\,u$$

Matrix Exponential

$$e^{A\,t} = \sum_{k=0}^{\infty} \frac{A\,t}{k!} = I + A\,t + \frac{A^2\,t^2}{2!} + \frac{A^3\,t^3}{3!} + \cdots$$

Laplace Transform of the Matrix Exponential

$$e^{A\,t} \circ\!\!-\!\!\bullet\ (s\,I - A)^{-1}$$

Solution of the State Equation: Example

Computation

Gap 1

Let $A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow e^{At} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -2t & 0 \\ 0 & t \end{bmatrix} + \begin{bmatrix} \frac{4t^2}{2!} & 0 \\ 0 & \frac{t^2}{2!} \end{bmatrix} + \ldots = \begin{bmatrix} 1 + \frac{-2t}{1!} + \frac{(-2t)^2}{2!} + \frac{(-2t)^3}{3!} + \ldots & 0 \\ 0 & 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} \end{bmatrix}$

$= \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{bmatrix}$

$e^{At} \circ\!\!-\!\!\bullet \left(\begin{bmatrix} s+2 & 0 \\ 0 & s-1 \end{bmatrix} \right)^{-1} = \frac{\begin{bmatrix} s-1 & 0 \\ 0 & s+2 \end{bmatrix}}{(s+2)(s-1)} = \begin{bmatrix} \frac{1}{s+2} & 0 \\ 0 & \frac{1}{s-1} \end{bmatrix}$

Solution of the State Equation: Derivation

Laplace Transform of the State Equation

$$s\,X(s) - x(0) = A\,X(s) + B\,U(s)$$

Derivation

Gap 2

$$\Rightarrow (sI - A)\,X(s) = x(0) + b\,U(s)$$

$$\Rightarrow X(s) = (sI-A)^{-1} x(0) + (sI-A)^{-1} b\,U(s)$$

$$x(t) = e^{-At} x(0) + \int_0^t e^{-A(t-\tau)}\, b\, u(\tau)\, d\tau$$

Solution of the State Equation: Solution

Solution for the State

$$x(t) = e^{A\,t}\,x_0 + \int_0^t e^{A\,(t-\tau)}\,b\,u(\tau)d\tau$$

Solution for the Output

$$y(t) = c^T x + d\,u = c^T e^{A\,t}\,x_0 + c^T \int_0^t e^{A\,(t-\tau)}\,b\,u(\tau)d\tau + d\,u$$

Parts of the Solution

- Zero-input solution ($u \equiv 0$): $y(t) = c^T e^{A\,t}\,x_0$
  → Solution of the state equation if no input is applied
- Zero-state solution ($x_0 = 0$): $y(t) = c^T \int_0^t e^{A\,(t-\tau)}\,b\,u(\tau)d\tau + d\,u$
  → Solution of the state equation if the initial condition is zero
  → Zero-state solution corresponds to transfer function

Solution of the State Equation: Transfer Function

Computation

Gap 3

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u, \quad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$y = (0 \;\; 1) x + 2u$$

$$\Rightarrow y(t) = [0 \;\; 1] \left(\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{t} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-2(t-\tau)} & 0 \\ 0 & e^{t-\tau} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} u(\tau) d\tau \right)$$

$$+ 2u = e^{t} + \int_0^t \left(2e^{-2(t-\tau)} + e^{t-\tau}\right) u(\tau)\, d\tau + 2u(t)$$

Input/Output Behavior

- Characterized by zero-state solution of state equation
- Equivalent computation from transfer function

$\Rightarrow$ We study input/output behavior of LTI systems using transfer functions

Rational Transfer Function: Basics

Transfer Function

$$G(s) = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots a_n s^n} = \frac{B(s)}{A(s)}$$

Notation

- Numerator degree: m
- Denominator degree: n
  n is called the *order* of the transfer function
- relative degree: $r = n - m$

Classification

- $r < 0$: Transfer function is improper
- $r > 0$: Transfer function is strictly proper
- $r \geq 0$: Transfer function is proper

Rational Transfer Function: Example

Computation

Gap 4

$G_1(s) = \frac{2s+1}{s^2+4s+4}$; $r = 2-1 = 1$ => strictly proper
order $n = 2$

$G_2(s) = \frac{4s^2+2}{s+8}$; $r = 1-2 = -1$ => improper
order $n = 1$

$G_3(s) = \frac{1}{2s^2+4s-8}$; $r = 2-0 = 2$ => strictly proper
order $n = 2$

Rational Transfer Function: Pole-Zero Representation

Rational Transfer Function

$$G(s) = \frac{b_0 + b_1 s + \cdots + b_m s^m}{a_0 + a_1 s + \cdots a_n s^n} = \frac{B(s)}{A(s)}$$

Fact

- A polynomial with degree n has n zeros
  $\rightarrow$ The numerator of $G(s)$ has m zeros $z_1, z_2, \ldots, z_m \in \mathbb{C}$. These zeros are called transfer function *zeros*
  $\rightarrow$ The denominator of $G(s)$ has n zeros $p_1, p_2, \ldots, p_n \in \mathbb{C}$. These zeros are called transfer function *poles*

Pole-zero Representation of the Transfer Function

$$G(s) = K \cdot \frac{(s - z_1) \cdots (s - z_m)}{(s - p_1) \cdots (s - p_n)} \text{ with } K = \frac{b_m}{a_n}$$

Rational Transfer Function: Example

Computation

Gap 5

G_1: zeros: $s = -1/2$; poles: $s = -2$ (double)

$$\Rightarrow G_1(s) = \frac{s + \frac{1}{2}}{(s+2)^2} \cdot 2$$

G_2: zeros: $s = 0$, $s = -1/2$; poles: $s = -8$

$$\Rightarrow G_2(s) = \frac{s(s + 1/2)}{s+8} \cdot 4$$

$G_3(s) = \frac{1}{2} \cdot \frac{1}{s^2 + 2s - 3}$; zeros: none; poles: $s_1 = 1$, $s_2 = -3$

$$G_3(s) = \frac{1}{2} \cdot \frac{1}{(s+3)(s-1)}$$

Rational Transfer Function: Pole-Zero Diagram

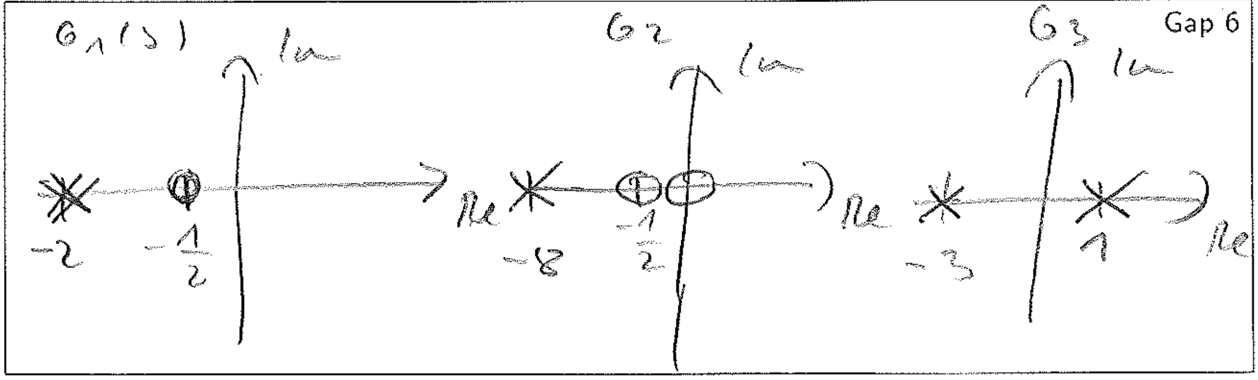
Fact

- If z_j (p_j) is a complex zero (pole) of $G(s)$, then the conjugated complex number z_j^* (p_j^*) is also a zero (pole) of $G(s)$

Pole-Zero Diagram

- Pole locations in the complex plane represented by crosses
- Zero locations in the complex plane represented by circles

Gap 6



BIBO Stability: Definition

Bounded Input Bounded Output (BIBO) Stability

A linear system with the transfer function $G(s)$ is called bounded input bounded output (BIBO) stable if for any bounded input u ($|u(t)| \leq u_{\max} < \infty$), the output y is also bounded ($|y(t)| \leq y_{\max} < \infty$).

$\Rightarrow$ In practice, we want systems to be BIBO stable!

Step Response Computation

Gap 7

Example, $G(s) = \dfrac{K}{(s+a)((s+b)^2+c^2)}$

$$\Rightarrow Y(s) = G(s)\cdot\frac{1}{s} = \frac{K}{(s+a)((s+b)^2+c^2)\cdot s} =$$

$$= \frac{A}{s+a} + \frac{Bs+C}{(s+b)^2+c^2} + \frac{D}{s} =$$

BIBO Stability: Computation

Step Response Computation

Gap 8

$$= \frac{A}{s+a} + B\frac{s+b}{(s+b)^2+c^2} + \frac{c}{(s+b)^2+c^2}\cdot\frac{C-Bb}{c} + \frac{D}{s}$$

$$\bullet\!\!-\!\!\circ\ \sigma(t)\left(e^{-at} + B e^{-bt}\sin ct + \frac{C-Bb}{c}e^{-bt}\cos ct + 1\right)$$

$\Rightarrow$ convergence for $a > 0$ and $b > 0$

poles of $G(s)$: $s_1 = -a$; $s_{2,3} = -b \pm jc$

Conclusion

- Step response for $G(s)$ with distinct poles remains finite if all poles of $G(s)$ lie in the open left half complex plane (OLHP)
- Step response for $G(s)$ with distinct poles becomes infinite if at least one pole of $G(s)$ lies in the right half plane (RHP)

BIBO Stability: Condition

General Stability Condition for Pole Locations

A linear system with the transfer function $G(s)$ is BIBO stable if and only if all poles of G are in the open left-half plane

Example

Gap 9

$G_1(s)$: $s_1 = s_2 = -2$ in OLHP $\Rightarrow$ stable

$G_2(s)$: $s_1 = -8$ in OLHP $\Rightarrow$ stable

$G_3(s)$: $s_1 = -3$, $s_2 = 1$: s_2 in RHP

$\Rightarrow$ instable !

BIBO Stability: Example

DC-Motor and Magnetic Suspension

Gap 10

DC-Motor: $G(s) \approx \frac{8.3 \cdot 10^6}{(s+85)(s+5000)}$

$p_1 = 85$, $p_2 = 5000$ in OLHP $\Rightarrow$ stable

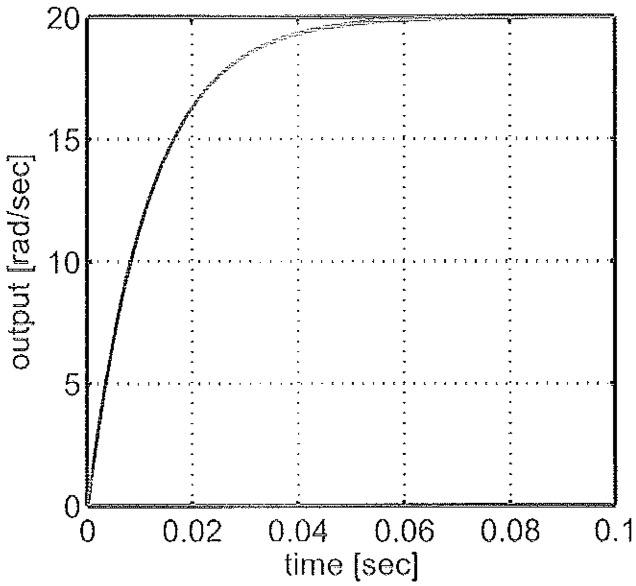
Magnetic suspension: $G(s) = \frac{K_i/m}{-s^2 + K_x/m}$

$\Rightarrow p_{1/2} = \pm\sqrt{\frac{K_x}{m}}$

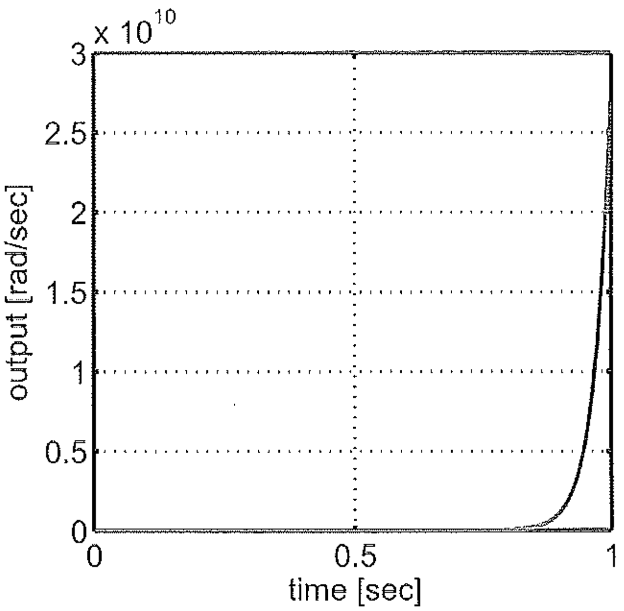
$\Rightarrow$ one pole in RHP $\Rightarrow$ instable !

BIBO Stability: Step Response Simulation

DC-Motor



Magnetic Suspension



BIBO Stability: Relation to Impulse Response

Condition for Impulse Response

If $g(t) \circ\!\!-\!\!\bullet\, G(s)$ is the impulse response of a linear system, then it is BIBO stable if and only if g is absolutely integrable: $\int_0^\infty |g(\tau)| d\tau < \infty$

Computation

Gap 11

We know that

$y(t) = g(t) * u(t) = \int_0^\infty g(\tau)\, u(t-\tau)\, d\tau \le$

$\le \int_0^\infty |g(\tau)| \cdot u_{max}\, d\tau = u_{max} \int_0^t |g(\tau)|\, d\tau < \infty$

only if g is absolutely integrable !

Stability Test: Routh-Hurwitz Method

Goal

- Decide about stability of a given transfer function

$$G(s) = \frac{B(s)}{A(s)} = \frac{B(s)}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$\Rightarrow$ Determine if $G(s)$ has poles in the right half plane

Routh-Hurwitz Method

- Finds out how many zeros of a polynomial $a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ are in the right half plane
- Does not compute the zeros explicitly

Stability Test: Routh-Array

Construction

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	b_1	b_2	b_3	$\cdots$	$\cdots$
s^{n-3}	c_1	c_2	c_3	$\cdots$	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$
s^1	w_1	0	0	0	$\cdots$
s^0	z_1	0	0	0	$\cdots$

Coefficients

- $b_1 = \dfrac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$
- $b_2 = \dfrac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$
- $b_3 = \dfrac{a_{n-1}a_{n-6} - a_n a_{n-7}}{a_{n-1}}$
- $c_1 = \dfrac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$
- $c_2 = \dfrac{b_1 a_{n-5} - a_{n-1} b3}{b_1}$
- $c_3 = \dfrac{b_1 a_{n-7} - a_{n-1} b_4}{b_1}$
- etc.

Stability Test: Example

Computation

Gap 12

$A(s) = s^5 + s^4 + 2s^3 + s^2 + s + 1$

$b_1 = \frac{2-1}{1} = 1$

$b_2 = \frac{1-1}{1} = 0$

$c_1 = \frac{1-0}{1} = 1$

$c_2 = \frac{1-0}{1} = 1$

$d_1 = \frac{-1}{1} = -1$

$d_2 = 0$

$e_1 = \frac{-1}{-1} = 1$

s^5	1	2	1
s^4	1	1	1
s^3	1	0	
s^2	1	1	
s^1	-1	0	
s^0	1	0	

$\Rightarrow$ two sign changes mean two instable poles

Stability Test: Routh-Hurwitz Criterion

Statement

Consider the first column of the Routh Array and call N_{diff} the number of sign changes (+/- or -/+) of the coefficients in that column. Then, N_{diff} is the number of zeros of the polynomial $A(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$ in the ORHP. That is, if $N_{\text{diff}} = 0$, then $A(s)$ has only zeros in the OLHP.

Special Cases

- If one coefficient in the first column of the Routh Array is 0, then either there are conjugated complex poles on the imaginary axis or there is at least one zero of $A(s)$ in the ORHP.
  $\rightarrow$ More details can be found in Ogata's book, Chapter 5-7

Stability Test

- Check the zeros of the denominator polynomial $A(s)$ of $G(s)$

Stability Test: Example

Computation

Gap 13

$A(s) = s^4 + 5s^3 + 5s^2 - 5s - 6$

$b_1 = \frac{25+5}{5} = 6$

$b_2 = \frac{-30}{5} = -6$

$c_1 = \frac{-30+30}{6} = 0$

⇒ 0 in first column

⇒ either zero or imag. axis or in RHP

⇒ instable

s^4	1	5	-6
s^3	5	-5	
s^2	6	-6	
s^1	0		
s^0			

Stability Test: Applicable Rules for Stability

Second-order Polynomial

$$A(s) = a_2 s^2 + a_1 s + a_0$$

⇒ a_0, a_1 and a_2 must have the same sign

Third-order Polynomial

$$A(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0$$

⇒ a_0, a_2, a_3 and $\frac{a_1 a_2 - a_0 a_3}{a_3}$ must have the same sign

Computation

Gap 14

$b_1 = \frac{a_1 a_0}{a_1} = a_0$

s^2	a_2	a_0
s^1	a_1	
s^0	a_0	

$b_1 = \frac{a_2 a_1 - a_3 a_0}{a_2}$

$c_1 = \frac{b_1 a_0}{b_1} = a_0$

s^3	a_3	a_1
s^2	a_2	a_0
s^1	b_1	0
s^0		