ECE 488 – Automatic Control

State-Space Models – Nonlinear Modeling

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Compulsory Course in Electronic and Communication Engineering
Credits (3/0/3)

Course Webpage: http://ECE488.cankaya.edu.tr

Reminder

Previous Topics

- Linear system modeling
- Block diagram representation
- Transfer functions
- Block diagram simplification

This Week

- State space models as alternative/equivalent system representation
- Nonlinear system modeling
- Set-point linearization of nonlinear state space models

Linear State Space Models: Definitions

Previously

- Study of dynamic relation between input (u) and output (y) signals

System State

The state $x(t_0)$ of a dynamic system at time t_0 is the information at time t_0 that is needed together with the input signal $u(t)$ for $t \geq t_0$ to determine the output signal $y(t)$ for $t \geq t_0$

Example RC-Circuit

Gap 1

$u(t) = \frac{y(t)}{R} + C\frac{dy(t)}{dt} \circ\!\!-\!\!\bullet\; U(s) = \frac{Y(s)}{R} + CsY(s) - Cy(0)$

$\Rightarrow Y(s) = \frac{1}{C}\cdot\frac{1}{s+\frac{1}{RC}}\cdot U(s) + \frac{y(0)}{s+\frac{1}{RC}} \;\bullet\!\!-\!\!\circ\; y(t) = \frac{1}{C}\cdot e^{-\frac{t}{RC}} * u(t)$

$+ y(0)e^{-\frac{t}{RC}} = y(0)e^{-\frac{t}{RC}} + \int_0^t e^{-\frac{(t-\tau)}{RC}}\cdot u(\tau)\,d\tau$

$\Rightarrow$ choose $y(t)$ as state

Linear State Space Models: Definitions

State Space Equations

$$\dot{x}(t) = A\,x(t) + b\,u(t) + o\,w(t)$$
$$y(t) = c^T x(t) + d\,u(t)$$

Signals

- State vector: $x(t) \in \mathbb{R}^n$
- State vector derivative: $\dot{x}(t) \in \mathbb{R}^n$
- Input: $u(t) \in \mathbb{R}$
- Output: $y(t) \in \mathbb{R}$
- Disturbance: $w(t) \in \mathbb{R}$

Constant Matrices and Vectors

- Dynamics matrix: $A \in \mathbb{R}^{n\times n}$
- Input vector: $b \in \mathbb{R}^{n\times 1}$
- Disturbance vector: $o \in \mathbb{R}^{n\times 1}$
- Output vector: $c^T \in \mathbb{R}^{1\times n}$
- Feed-through: $d \in \mathbb{R}$

Linear State Space Models: RLC-Circuit Example

Equations

Gap 2

$$\frac{d^2y(t)}{dt^2} = -\frac{1}{RC}\frac{dy}{dt} - \frac{1}{LC}y + \frac{1}{C}\frac{du}{dt}$$

Choose $x_1 = y$

$$x_2 = \dot{y} - \frac{1}{C}u = \dot{x}_1 - \frac{1}{C}u$$

$$\Rightarrow \dot{x}_1 = \dot{y} = x_2 + \frac{1}{C}u$$

$$\dot{x}_2 = \ddot{y} - \frac{1}{C}\dot{u} = -\frac{1}{RC}\dot{y} - \frac{1}{LC}y + \cancel{\frac{1}{C}\dot{u}} - \cancel{\frac{1}{C}\dot{u}}$$

$$= -\frac{1}{RC}\left(x_2 + \frac{1}{C}u\right) - \frac{1}{LC}x_1$$

$$\Rightarrow \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{1}{LC} & -\frac{1}{RC}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}\frac{1}{C}\\ -\frac{1}{RC^2}\end{bmatrix}u; \quad y = \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

Linear State Space Models: DC-Motor Example

Equations

Gap 3

$$\frac{di_a}{dt} = \frac{1}{L_a}u - \frac{R_a}{L_a}i_a - \frac{c\Phi_F}{L_a}\omega \quad \Rightarrow \text{input: } u$$

$$\frac{d\omega}{dt} = \frac{1}{J}c\Phi_F i_a - \frac{1}{J}T_L \quad \Rightarrow \text{state: } i_a, \omega$$

$$\Rightarrow \text{output: } y = \omega$$

$$\begin{bmatrix}\frac{di_a}{dt}\\ \frac{d\omega}{dt}\end{bmatrix} = \begin{bmatrix}-\frac{R_a}{L_a} & -\frac{c\Phi_F}{L_a}\\ \frac{c\Phi_F}{J} & 0\end{bmatrix}\begin{bmatrix}i_a\\ \omega\end{bmatrix} + \begin{bmatrix}\frac{1}{L_a}\\ 0\end{bmatrix}u + \begin{bmatrix}0\\ -\frac{1}{J}\end{bmatrix}T_L$$

$$y = \begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}i_a\\ \omega\end{bmatrix}$$

Linear State Space Models: Relation to Transfer Function

Linear State Space Model **Laplace Transformation**

$\dot{x} = A \cdot x + b \cdot u + o \cdot w$ $\quad sX(s) - x(0) = A \cdot X(s) + b \cdot U(s) + o \cdot W(s)$

$y = c^T \cdot x + d \cdot u$ $\quad Y(s) = c^T \cdot X(s) + d \cdot U(s)$

Computation for $x(0) = 0$

Gap 4

$$(sI - A) X(s) = b U(s) + o W(s)$$

$$\Rightarrow X(s) = (sI - A)^{-1} b U(s) + (sI - A)^{-1} o W(s)$$

$$\Rightarrow Y(s) = c^T X(s) + d U(s) = [c^T (sI - A)^{-1} b + d] U(s) + c^T (sI - A)^{-1} o W(s)$$

$$\Rightarrow Y(s) = \underbrace{(c^T (sI - A)^{-1} b + d)}_{\text{Plant transfer function } G(s)} U(s) + \underbrace{c^T (sI - A)^{-1} o}_{\text{Disturbance transfer function } G_d(s)} W(s)$$

Linear State Space Models: RLC-Circuit Transfer Function

Computation

Gap 5

$$G(s) = [1 \quad 0] \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -\frac{1}{LC} & -\frac{1}{RC} \end{bmatrix} \right)^{-1} \begin{bmatrix} \frac{1}{C} \\ -\frac{1}{RC^2} \end{bmatrix} =$$

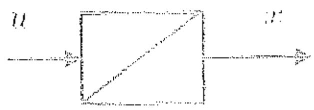
$$= [1 \quad 0] \begin{bmatrix} s & -1 \\ +\frac{1}{LC} & s + \frac{1}{RC} \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{C} \\ -\frac{1}{RC^2} \end{bmatrix} =$$

$$= [1 \quad 0] \frac{\begin{bmatrix} s + \frac{1}{RC} & 1 \\ * & * \end{bmatrix}}{s^2 + \frac{1}{RC} s + \frac{1}{LC}} \begin{bmatrix} \frac{1}{C} \\ -\frac{1}{RC^2} \end{bmatrix} = \frac{\frac{1}{C} s + \frac{1}{RC^2} - \frac{1}{RC^2}}{s^2 + \frac{1}{RC} s + \frac{1}{LC}} =$$

$$= \frac{Ls}{1 + \frac{L}{R} s + LC s^2}$$

Linear State Space Models: Relation to Block Diagram

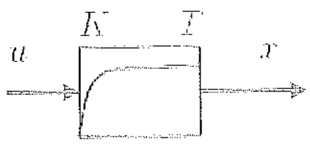
Integrator



State Space Model

- State: x (integrator output)
- State equation: $\dot{x} = u$

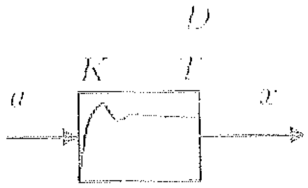
First-order Lag



State Space Model

- State: x (first-order lag output)
- State equation: $\dot{x} = \frac{1}{T}(K \cdot u - x)$

Second-order Lag



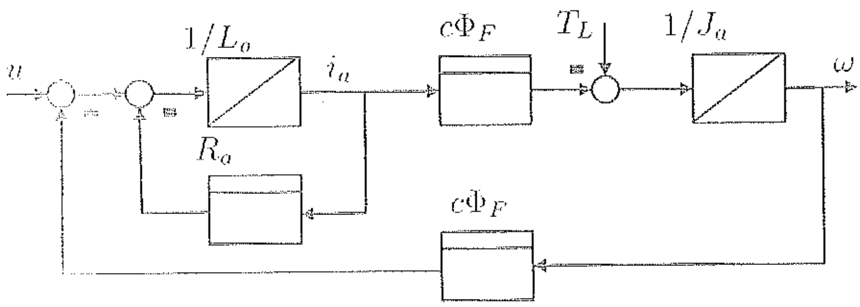
State Space Model

- States: x, $\hat{x}$
- State equation:

$$\dot{x} = \hat{x}$$

$$\dot{\hat{x}} = \frac{1}{T^2}(K \cdot u - x - 2\,D\,T\,\hat{x})$$

Linear State Space Models: DC-Motor Example



Computation

Gap 6

$x_1 = i_a$; $x_2 = \omega$; $w = T_L$

$$\frac{dx_1}{dt} = \frac{1}{L_a}(-R_a x_1 - c\Phi_F x_2 + u)$$

$$\frac{dx_2}{dt} = \frac{1}{J_a}(c\Phi_F x_1 - w)$$

same as above !

$y = x_2$

Nonlinear System Modeling: Remarks

LTI System Operators

- Proportional gain
- Differentiation
- Integration
- Lead/lag components
- Summations

⇒ All linear operators can be represented by transfer functions

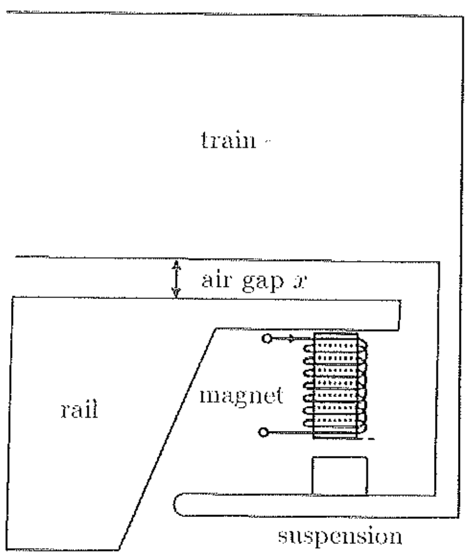
Nonlinear Systems

- Contain nonlinear system operators

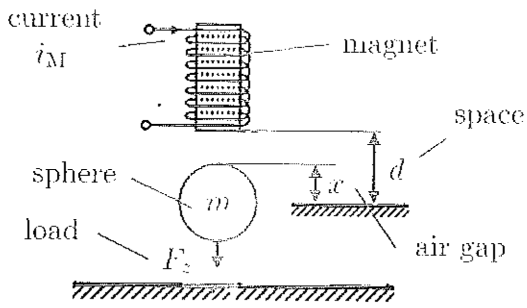
⇒ No transfer function representation

Nonlinear System Modeling: Magnetic Suspension

Schematic



Simplified Description



Simplifications

- Sphere represents vehicle
- Magnet represents suspension system

Nonlinear System Modeling: Equations

Computation

Gap 7

Magnetic Force: $F_m = K_m \frac{i_m^2}{(d-x)^2}$ (from textbook)

Force balance: $m\ddot{x} = F_m - F_z - mg = K_m \frac{i_m^2}{(d-x)^2} - F_z - mg$

Input: $u = i_m$; states: $x_1 = x$; $x_2 = \dot{x}$, output: $y = x$

disturbance: $w = F_z$

State Equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{1}{m}\left(K_m \frac{u^2}{(d-x_1)^2} - w - mg\right)$$

$$y = x_1$$

Nonlinear System Modeling

Nonlinear State Equations: General Form

State Equations

$$\dot{x} = f(x, u, w)$$
$$y = h(x, u)$$

Notation

- state: x, output: y, input: u, disturbance w
- f: continuous in x, u, w and additional assumptions (see for example ECE 564)
- h: continuous in x, u

Example

Gap 8

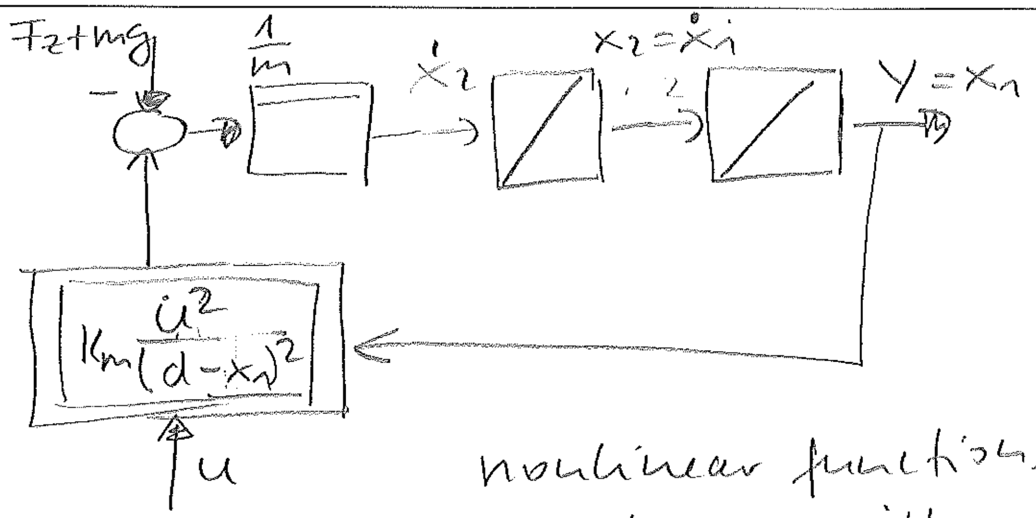
$$\dot{x} = \begin{bmatrix} x_2 \\ \frac{K_m}{m}\frac{u^2}{(d-x_1)^2} - \frac{w}{m} - g \end{bmatrix} = f(x, u, w)$$

$$y = x_1 = h(x, u)$$

Nonlinear State Equations: Block Diagram

Example

Gap 9



nonlinear functions are shown with a "double frame"

Nonlinear System Modeling: Remarks

Synthesis and Analysis Techniques for Nonlinear Systems

- Beyond the scope of this lecture
  → Master-level course ECE 564
- Extensive literature
  → Alberto Isidori: "Nonlinear Control Systems", Springer, 1995 (ISBN: 3-54-019916-0)
  → Hassan K. Khalil: "Nonlinear Systems", Prentice Hall, 2002 (ISBN: 0-13-067389-7)

Set-point Linearization

- Consider system behavior in the vicinity of a given set-point
  → Assume almost linear behavior close to the set-point
  → Find a linear system model to approximate the nonlinear system

Set-Point: Definition

Set-point Definition

A set point is a stationary (non-changing) state of a system where the system output maintains a constant set-point value y_{SP}

Computation of a Set-point

- Given: y_{SP}, w_{SP}
- We want to compute x_{SP} (constant set-point value of the state) and u_{SP} (constant set-point value of the input)
- Computation

$$y_{SP} = h(x_{SP}, u_{SP})$$
$$0 = \dot{x} = f(x_{SP}, u_{SP}, w_{SP})$$

$\Rightarrow$ Solve for x_{SP}, u_{SP}

Set-Point: Example

Magnetic Suspension

Gap 10

For example, we want the position at $y_{SP} = \frac{d}{2}$ and we assume no disturbance in the set-point $w_{SP} = 0$.

From $\dot{x} = 0$, $x_{2,SP} = 0$ and

$$\dot{x}_2 = 0 = K_m \frac{u_{SP}^2}{(d - x_{1,SP})^2} - g - w_{SP} = K_m \frac{u_{SP}^2}{\frac{d^2}{4}} - g$$

$$\Rightarrow u_{SP} = \frac{d}{2}\sqrt{\frac{mg}{K_m}}$$

Set-point: $x_{1,SP} = \frac{d}{2}$, $x_{2,SP} = 0$, $w_{SP} = 0$

$$u_{SP} = \frac{d}{2}\sqrt{\frac{mg}{K_m}}$$

Set-point Linearization: Description

Explanation

- Compute a „small signal" approximation of the nonlinear system that is valid close to the set-point
- Introduce „Difference variables" (deviation from the set-point)
- $\Delta x = x - x_{SP}$, $\Delta y = y - y_{SP}$, $\Delta u = u - u_{SP}$, $\Delta w = w - w_{SP}$

Taylor Series Expansion

$$\Delta\dot{x} = \dot{x} \approx \underbrace{f(x_{SP}, u_{SP}, w_{SP})}_{=0} + \underbrace{\frac{\partial f}{\partial x}|_{SP}}_{A} \Delta x + \underbrace{\frac{\partial f}{\partial u}|_{SP}}_{b} \Delta u + \underbrace{\frac{\partial f}{\partial w}|_{SP}}_{o} \Delta w$$

$$= A\,\Delta x + b\,\Delta u + o\,\Delta w$$

$$\Delta y \approx \underbrace{h(x_{SP}, u_{SP}) - y_{SP}}_{=0} + \underbrace{\frac{\partial h}{\partial x}|_{SP}}_{c^T} \Delta x + \underbrace{\frac{\partial f}{\partial u}|_{SP}}_{d} \Delta u = c^T\,\Delta x + d\,\Delta u$$

Set-point Linearization: Example

Example Equations

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -g + \frac{K_M}{m}\frac{u^2}{(d-x_1)^2} - \frac{1}{m}w$$

$$y = x_1$$

Computation

Gap 11

$$A = \frac{\partial f}{\partial x}|_{SP} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}_{SP} =$$

$$= \begin{bmatrix} 0 & 1 \\ \frac{K_m u^2 \cdot 2}{m(d-x_1)^3} & 0 \end{bmatrix}_{SP} =$$

Set-point Linearization: Example

Computation

Gap 12

$$= \begin{bmatrix} 0 & 1 \\ \dfrac{2\cancel{K_m}}{\cancel{m}}\dfrac{\frac{d^2}{4}\frac{\cancel{m}g}{\cancel{K_m}}}{(d/2)^3} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \dfrac{4g}{d} & 0 \end{bmatrix}$$

$$b = \frac{\partial f}{\partial u}\Big|_{sp} = \begin{bmatrix} \partial f_1/\partial u \\ \partial f_2/\partial u \end{bmatrix}_{sp} = \begin{bmatrix} 0 \\ \dfrac{2K_m}{m}\dfrac{u}{(d-x_1)^2} \end{bmatrix}_{sp} = \begin{bmatrix} 0 \\ \dfrac{4}{d}\sqrt{\dfrac{K_m g}{m}} \end{bmatrix}$$

$$\sigma = \frac{\partial f}{\partial w}\Big|_{sp} = \begin{bmatrix} \partial f_1/\partial w \\ \partial f_2/\partial w \end{bmatrix}_{sp} = \begin{bmatrix} 0 \\ -\frac{1}{m} \end{bmatrix}; \quad c^T = \frac{\partial h}{\partial x}\Big|_{sp} = [1 \;\; 0]$$

$$\Rightarrow \Delta\dot{x} = \begin{bmatrix} 0 & 1 \\ \frac{4g}{d} & 0 \end{bmatrix}\Delta x + \begin{bmatrix} 0 \\ \frac{4}{d}\sqrt{\frac{K_m \cdot g}{m}} \end{bmatrix}\Delta u + \begin{bmatrix} 0 \\ -1/m \end{bmatrix}\Delta w$$

$$\Delta y = [1 \;\; 0]\Delta x$$

Set-point Linearization: Illustration

Tangent Approximation

- Compute slope of nonlinear function in set-point
  → Replace nonlinear function by its tangent at set-point

Illustration

Gap 13

Example $y = \frac{1}{4}u^2$

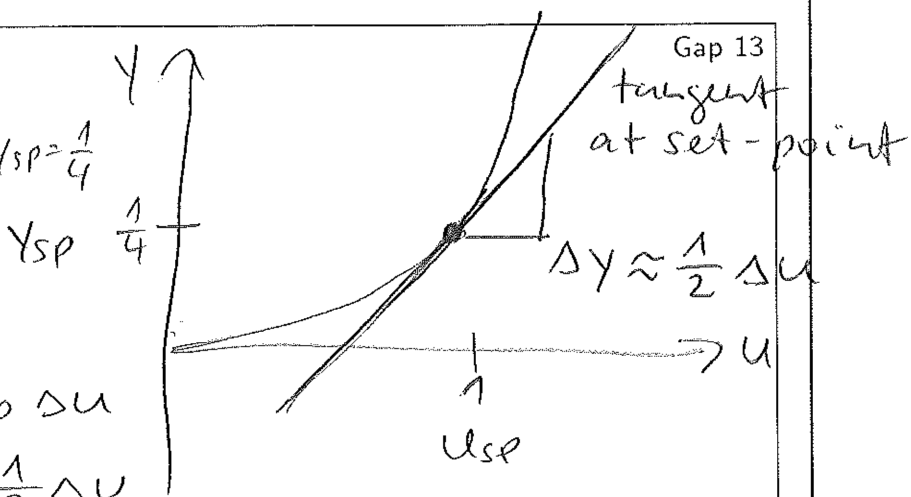
set-point $u_{sp} = 1$, $y_{sp} = \frac{1}{4}$

$\to \Delta y = y - \frac{1}{4}$

$\Delta u = u - 1$

$\Delta y \approx \frac{\partial}{\partial u}\left(\frac{1}{4}u^2\right)\Big|_{sp}\Delta u$

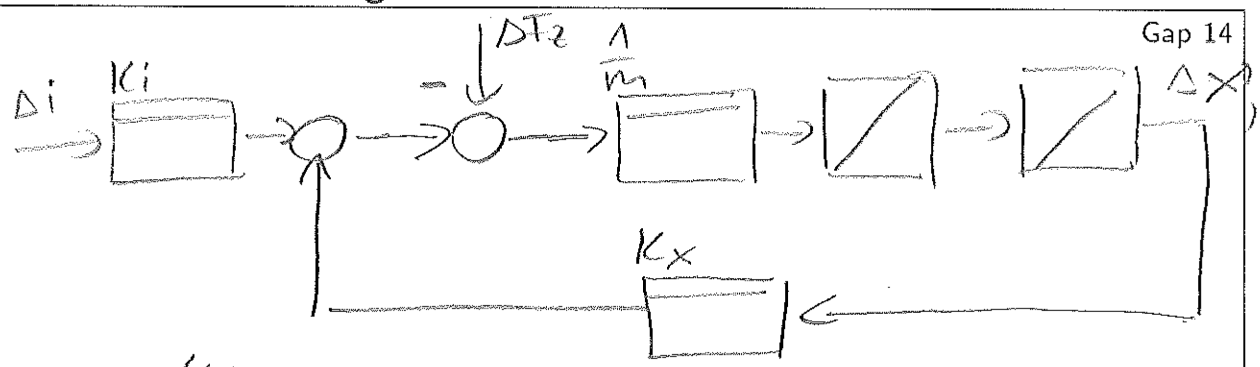
$= \frac{1}{2}u|_{sp}\Delta u = \frac{1}{2}\Delta u$



Set-point Linearization: Magnetic Suspension Example

Linearized Block Diagram

Gap 14



$K_x = \frac{4g}{d}$

$K_i = \frac{4}{d}\sqrt{\frac{K_m g}{m}}$

Linearization: Summary

Task

- Characterize nonlinear system behavior close to a set-point

Method

- Write system representation in terms of "difference variables"
- Use first-order Taylor series approximation for nonlinearities

Result

- We get a linear system model for the nonlinear system
- Linear methods can be used for the nonlinear system close to the set-point
- Important restriction
  → Linear model is only valid in the vicinity of the set-point