ECE 488 – Automatic Control

Transfer Functions – Block Diagram Simplification

Assistant Prof. Dr. Klaus Schmidt

Department of Mechatronics Engineeering – Çankaya University

Compulsory Course in Electronic and Communication Engineering
Credits (3/0/3)

Course Webpage: http://ECE488.cankaya.edu.tr

Reminder

Previous Topics

- Linear system modeling (DC-motor)
- Linear time-invariant ordinary differential equations
- Block diagram representation

This Week

- Transfer functions
- Block diagram simplification

Transfer Functions: Transfer Block Representation

System Operator S

- Mapping from input signal u to output signal y: $y(t) = S(u(t))$

Gap 1

Proportional Gain: $y(t) = K u(t)$; $S(u(t)) = K u(t)$

Integrator: $y(t) = \int_0^t u(\tau) d\tau$; $S(u(t)) = \int_0^t u(\tau) d\tau$

Differentiator: $y(t) = \frac{du(t)}{dt}$; $S(u(t)) = \frac{du(t)}{dt}$

Transfer Function $G(s)$

- Relation between Laplace transforms (uni-directional) of input signal $u(t)$ and output signal $y(t)$

Transfer Functions: Laplace Domain Representation

Laplace Transform

- Transformed input signal: $u(t) \circ\!\!-\!\!\bullet \mathcal{L}(u(t)) = U(s)$
- Transformed output signal: $y(t) \circ\!\!-\!\!\bullet \mathcal{L}(y(t)) = Y(s)$
- Laplace transform of the LTI ODE (zero initial conditions):

$$\mathcal{L}\big(a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 \dot{y} + a_0 y\big) = \mathcal{L}\big(b_m u^{(m)} + b_{m-1} u^{(m-1)} + \cdots + b_1 \dot{u} + b_0 u\big)$$

$$\left(\frac{d}{dt} u(t) \circ\!\!-\!\!\bullet sU(s) + u(0)\right) \quad \Downarrow \quad \left(\frac{d}{dt} y(t) \circ\!\!-\!\!\bullet sY(s) + y(0)\right)$$

Gap 2

We consider that $u(0) = \dot{u}(0) = \dots = u^{(n-1)} = 0$

and $y(0) = \dot{y}(0) = \dots = y^{(n-1)}(0) = 0$

$a_n s^n Y(s) + a_{n-1} s^{n-1} Y(s) + \dots + a_1 s Y(s) + a_0 Y(s) =$

Transfer Functions: Input/Output Relation

Gap 3

$$= b_m s^m U(s) + \dots + b_1 s\, U(s) + b_0 U(s)$$

$$\Rightarrow (a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0)\, Y(s) =$$

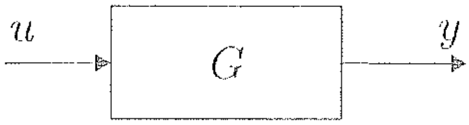
$$= (b_m s^m + \dots + b_1 s + b_0)\, U(s)$$

Input/Output Relation: Transfer Function $G(s)$

$$\frac{Y(s)}{U(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} =: G(s) = \frac{B(s)}{A(s)}$$

Transfer Block Representation

- Input/output variables u, y
- Dynamic relation G



Transfer Functions: Examples

RLC-Circuit

Gap 4

$$C \frac{d y^2}{dt^2} + \frac{1}{R} \frac{dy}{dt} + \frac{1}{L} y = \frac{du}{dt}$$

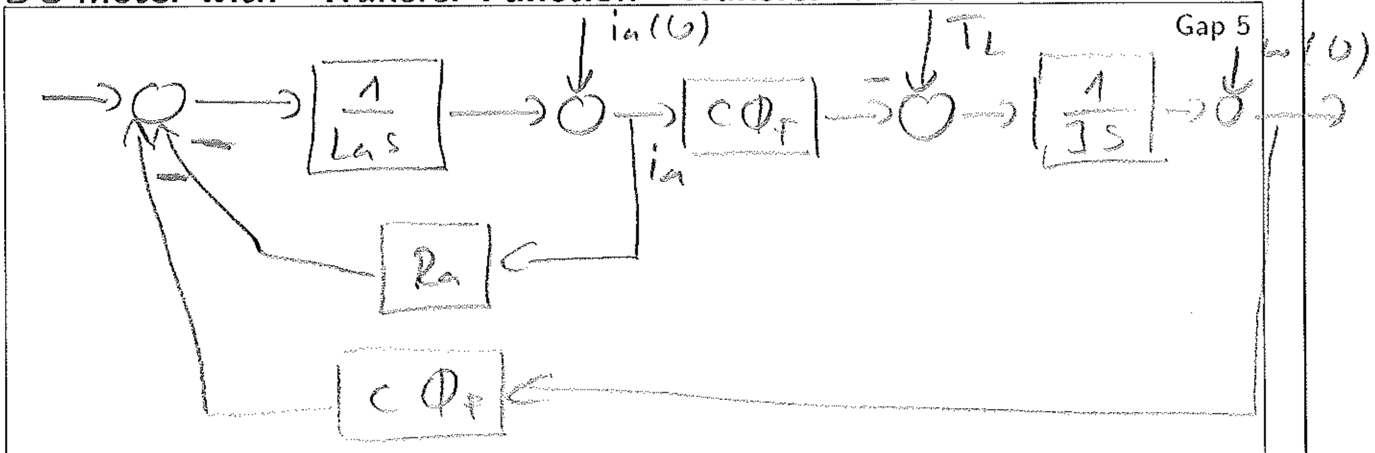
$$\Rightarrow \left(C s^2 + \frac{1}{R} s + \frac{1}{L}\right) Y(s) = s\, U(s)$$

$$\Rightarrow G(s) = \frac{s}{C s^2 + \frac{1}{R} s + \frac{1}{L}} = \frac{L s}{L C s^2 + \frac{L}{R} s + 1}$$

(second-order transfer function)

Transfer Functions: Example

DC-motor with "Transfer Function" Transfer Blocks



$\Rightarrow$ System operators "$\int$" are replaced by transfer functions $\frac{1}{s}$

Transfer Functions: Facts

Impulse Response

- Consider system operator S and dirac impulse $\delta(t)$ (recall that $y(t) = S(u(t))$)
  $\Rightarrow$ System reaction to Dirac impulse input signal $u(t) = \delta(t)$
  $$g(t) = S(\delta(t))$$
  $\Rightarrow$ It holds that $g(t) \circ\!\!-\!\!\bullet\ G(s)$

Output Computation

- Convolution: $g(t) \star u(t) = \int_{-\infty}^{\infty} g(t-\tau)\, u(\tau)\, d\tau$
  $$\Rightarrow y(t) = g(t) \star u(t)$$
- Inverse Laplace transform: $y(t) = \mathcal{L}^{-1}(Y(s))$
  $$\Rightarrow y(t) = g(t) \star u(t) = \mathcal{L}^{-1}(G(s)\, U(s)) = \mathcal{L}^{-1}(Y(s))$$

Step Responses: Computation

Definition

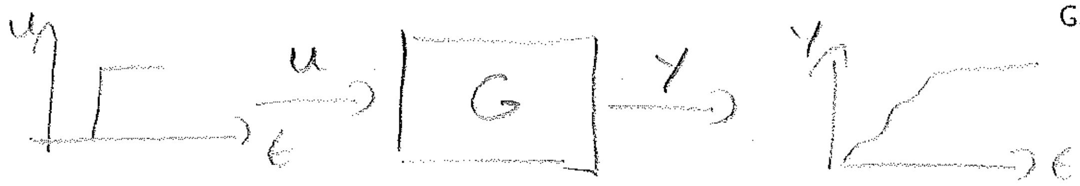
The unit step response is the output response of a dynamic system to the Heaviside step function applied at its input

- Heaviside step function: $\sigma(t) = \begin{cases} 0 & \text{if } t < 0 \\ 1 & \text{if } t \geq 0 \end{cases}$

Step Response Computation

- Input: $u(t) = \sigma(t) \circ\!\!-\!\!\bullet\ U(s) = \frac{1}{s}$
- Output: $Y(s) = G(s)U(s) = G(s)\frac{1}{s} \bullet\!\!-\!\!\circ\ y(t) = \mathcal{L}^{-1}(G(s)\frac{1}{s})$

Gap 6

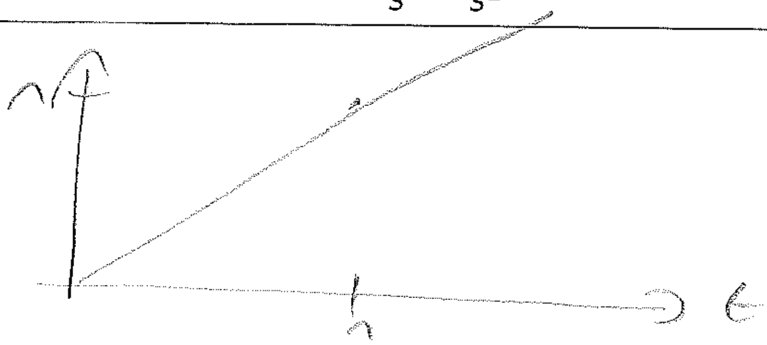


Step Responses: Integrator Example

Integrator

- ODE: $\dot{y} = u$
- Laplace transform: $sY(s) = U(s)$
- Transfer function: $G(s) = \frac{1}{s}$
- Step response: $Y(s) = G(s)\frac{1}{s} = \frac{1}{s^2} \bullet\!\!-\!\!\circ\ \sigma(t) \cdot t$

Gap 7



Step Responses: First-order Lag Example

First-order Lag

- ODE: $T \cdot \dot{y} + y = K \cdot u$
- Laplace transform:
- Transfer function:
- Step response:

Gap 8

$(sT+1)\,Y(s) = K \cdot U(s)$

$\Rightarrow G(s) = \dfrac{K}{1+sT}$

$y(t) = K\,\sigma(t)\,(1 - e^{-t/T})$

Gap 9

Partial Fractional Decomposition

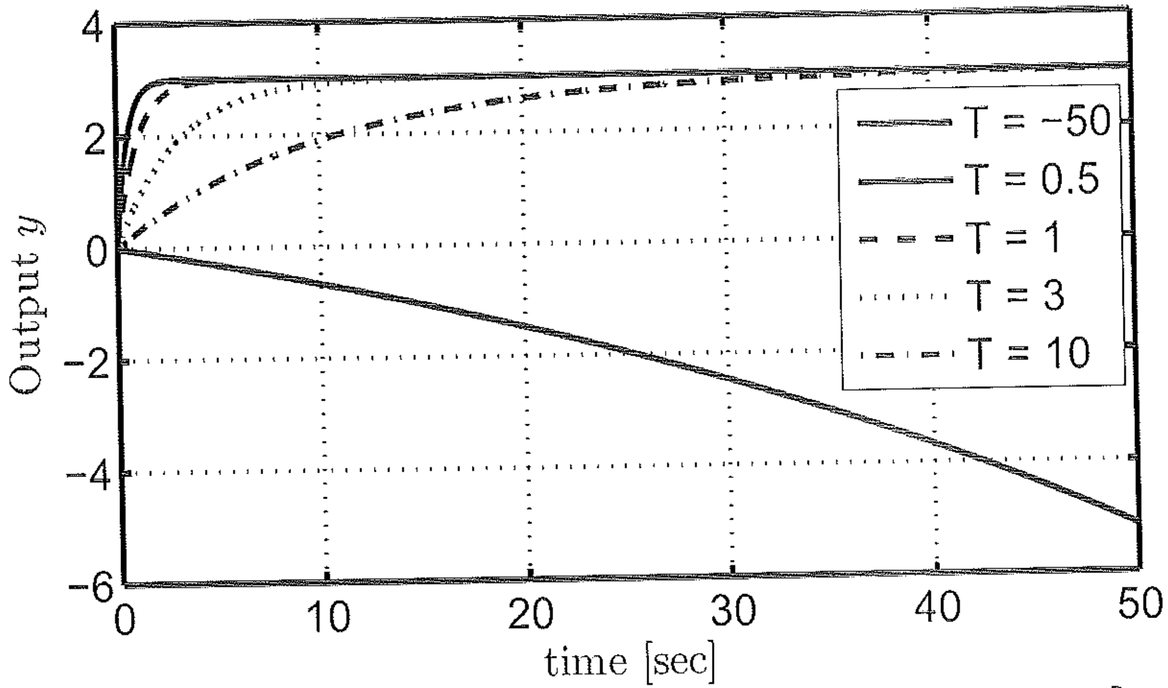
$Y(s) = G(s)\cdot U(s) = \dfrac{K}{1+sT}\cdot\dfrac{1}{s} = \dfrac{A}{1+sT} + \dfrac{B}{s} =$

$= \dfrac{As + B + BsT}{(1+sT)\cdot s} \Rightarrow K = B$ and $A + BT = 0 \Rightarrow A = -KT$

$\Rightarrow Y(s) = \dfrac{K}{s} - \dfrac{K}{s+\frac{1}{T}} \;\bullet\!\!-\!\!\circ\; K\sigma(t) - Ke^{-t/T}\sigma(t)$

Step Responses: First-order Lag Example

First-order Lag for Different T



Step Responses: Time Delay Example

Time Delay

- Input-output relation: $y(t) = u(t - \Delta)$
- Laplace transform:
- Transfer function:
- Step response:

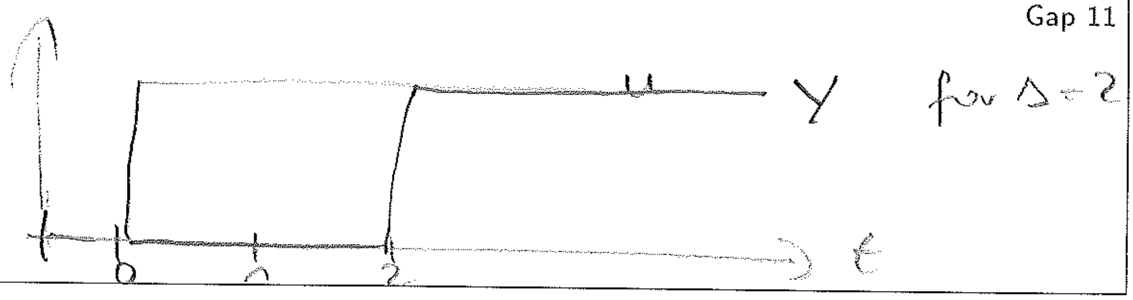
Gap 10

$Y(s) = U(s)\, e^{-s\Delta}$

$\Rightarrow G(s) = e^{-s\Delta}$

$y(t) = \sigma(t - \Delta)$

Gap 11



Step Responses: Second-order Lag Example

Second-order Lag

- ODE: $T^2 \cdot \ddot{y} + 2DT\dot{y} + y = K \cdot u$
- Laplace transform: $s^2 T^2 \cdot Y(s) + 2DTsY(s) + Y(s) = K \cdot U(s)$
- Transfer function: $G(s) = \dfrac{K}{1 + 2DT \cdot s + T^2 \cdot s^2}$
- Step response: Different solutions depending on damping constant D

Denominator Zeros

Gap 12

zeros of $T^2 s^2 + 2DTs + 1$

$\rightarrow s_{1/2} = \frac{1}{2T^2}\left(-2DT \pm \sqrt{4D^2T^2 - 4T^2}\right)$

Case 1: $D > 1$: $\sqrt{\cdot}$ is real $\Rightarrow$ two real zeros

Case 2: $D = 1$: $\sqrt{\cdot} = 0$ $\Rightarrow$ double zero at $s = -\frac{1}{T}$

Case 3: $D < 1$: $\sqrt{\cdot}$ is imaginary $\Rightarrow$ two conjugated complex zeros

Step Responses: Second-order Lag Example

Case 1: $D > 1$ (aperiodic case)

- $G(s) = \dfrac{K}{(1+T_1 s)(1+T_2 s)}$
- $y(t) = \sigma(t)K(1 - \frac{T_1}{T_1 - T_2}e^{-t/T_1} + \frac{T_2}{T_1-T_2}e^{-t/T_2})$

Case 2: $D = 1$ (aperiodic limit)

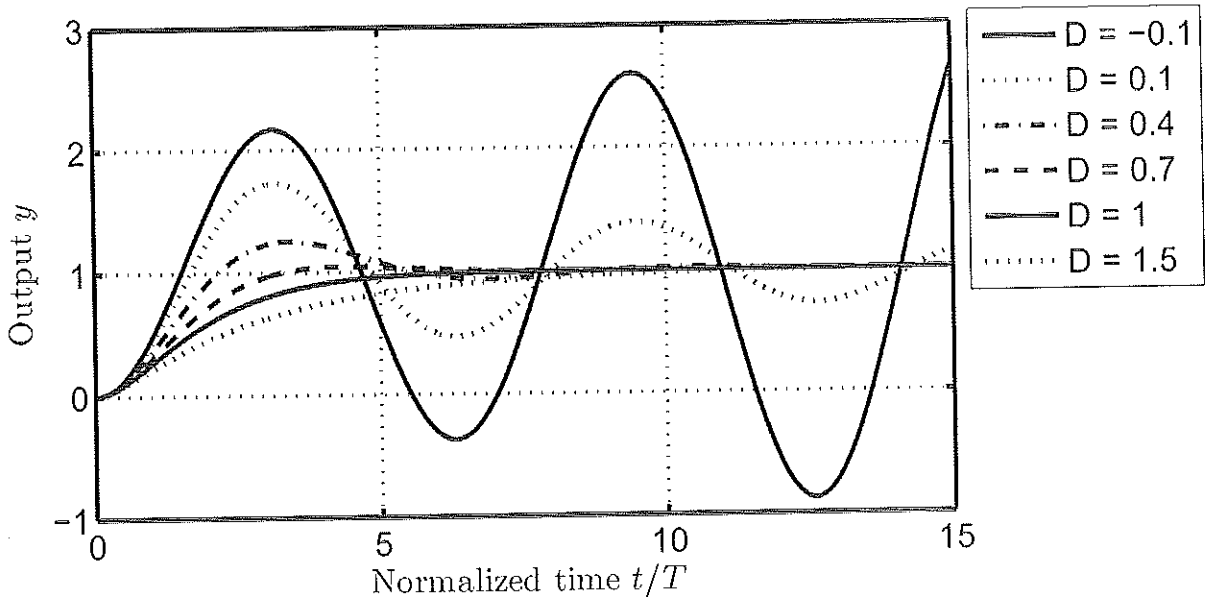
- $G(s) = \dfrac{K}{(1+Ts)^2}$
- $y(t) = \sigma(t)K\big(1 - (1+\frac{t}{T})e^{-t/T}\big)$

Case 3: $D < 1$ (periodic case)

- $G(s) = \dfrac{K}{(1+2DTs+T^2s^2)}$
- $y(t) = \sigma(t)K\big(1 - \frac{e^{-Dt/T}}{\sqrt{1-D^2}}\sin(\frac{\sqrt{1-D^2}}{T}t + \arctan\frac{\sqrt{1-D^2}}{D})\big)$

Step Responses: Second-order Lag Example

Second-order Lag for Different D



Block Diagrams: Standardized Transfer Blocks

Proportional Gain

$$y(t) = K\sigma(t)$$

Integrator

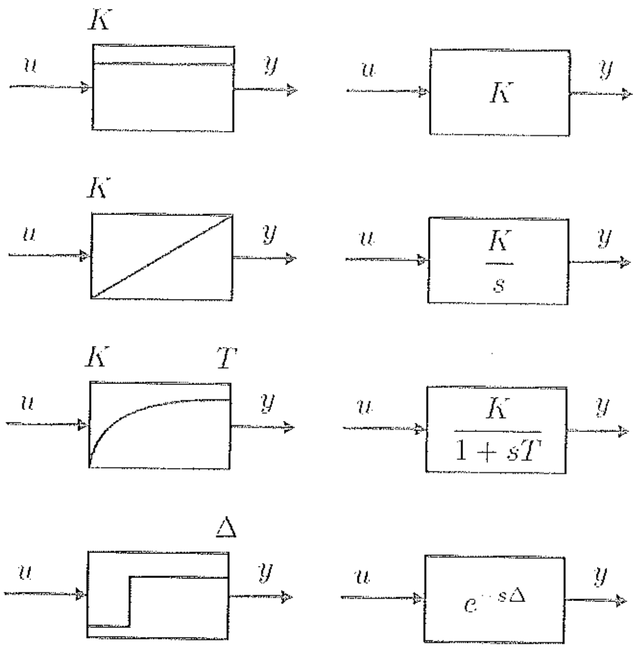
$$y(t) = K\sigma(t)t$$

First-order Lag

$$y(t) = K\sigma(t)(1 - e^{-t/T})$$

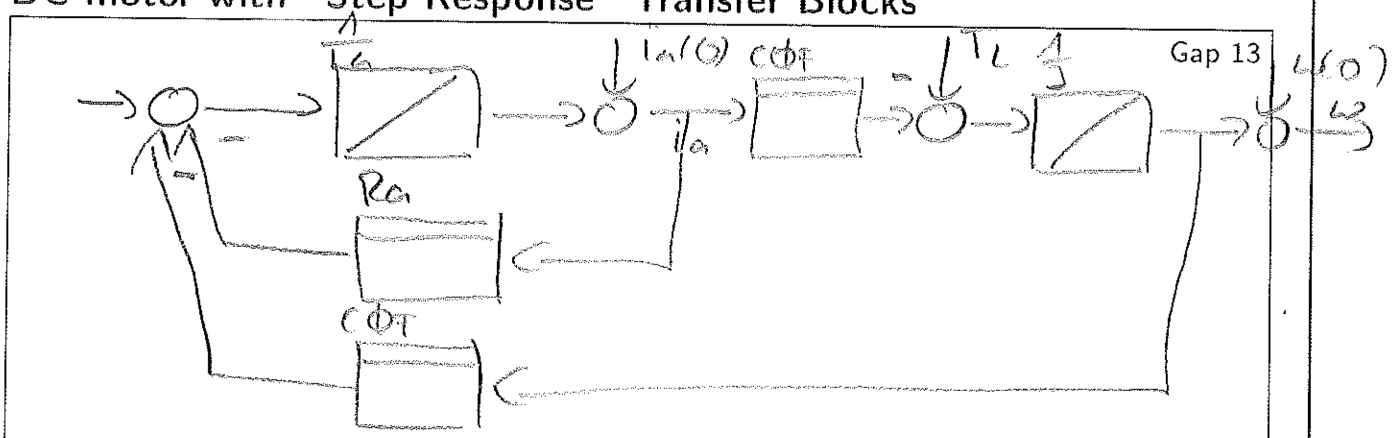
Time Delay

$$y(t) = \sigma(t - \Delta)$$



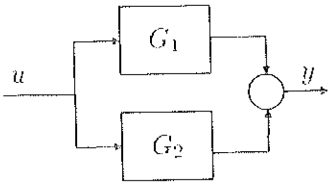
Plant Modeling: Example

DC-motor with "Step Response" Transfer Blocks

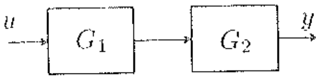


Block Diagram Simplification: Connection Rules

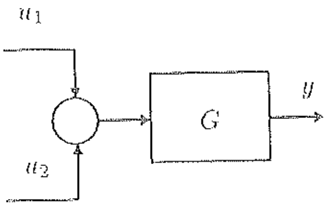
Parallel



Series



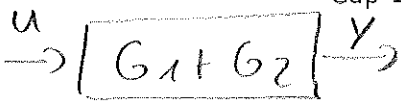
Summation 1



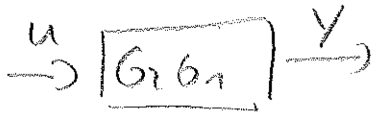
Equivalent Representation

Gap 14

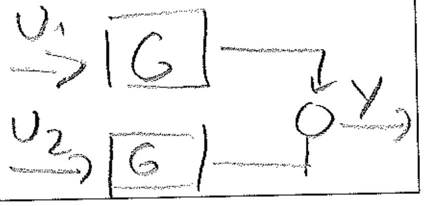
$Y = G_1U + G_2U = (G_1+G_2)U$



$Y = G_2(G_1U) = (G_2G_1)U$



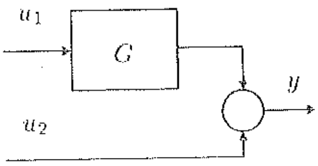
$Y = G(U_1+U_2) = GU_1 + GU_2$



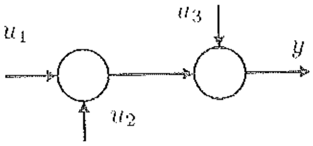
Klaus Schmidt
ECE 488 – Automatic Control
Department

Block Diagram Simplification: Connection Rules

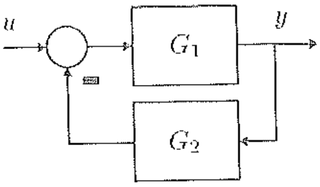
Summation 2



Summation 3



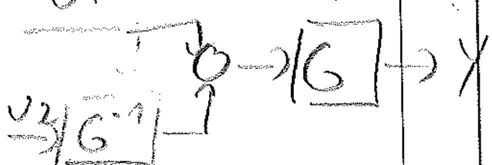
Feedback



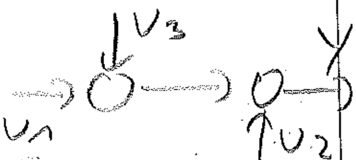
Equivalent Representation

Gap 15

$Y = GU_1 + U_2 = G(U_1 + \frac{1}{G}U_2)$



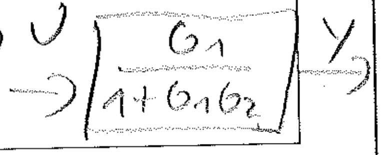
$Y = U_3 + (U_1+U_2) = (U_3+U_1) + U_2$



$Y = G_1(U - G_2Y)$

$\Rightarrow Y(1+G_1G_2) = G_1U$

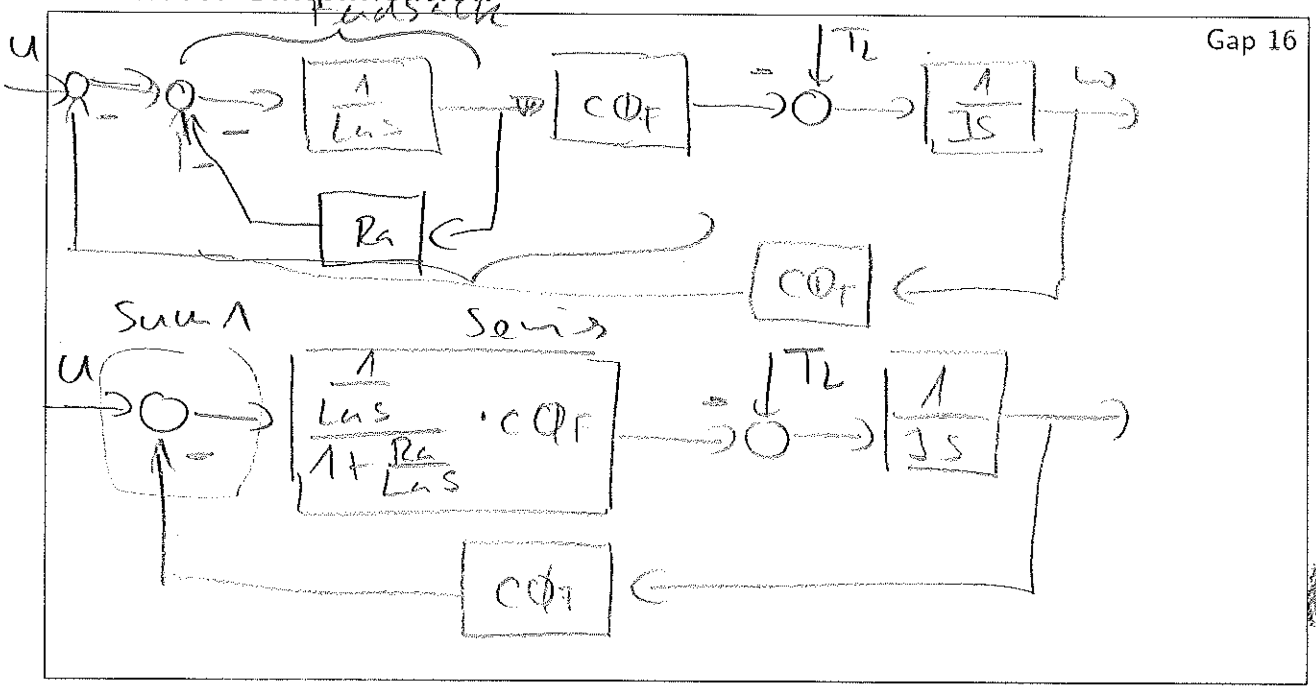
$\Rightarrow Y = \frac{G_1}{1+G_1G_2}U$



Klaus Schmidt
ECE 488 – Automatic Control
Department

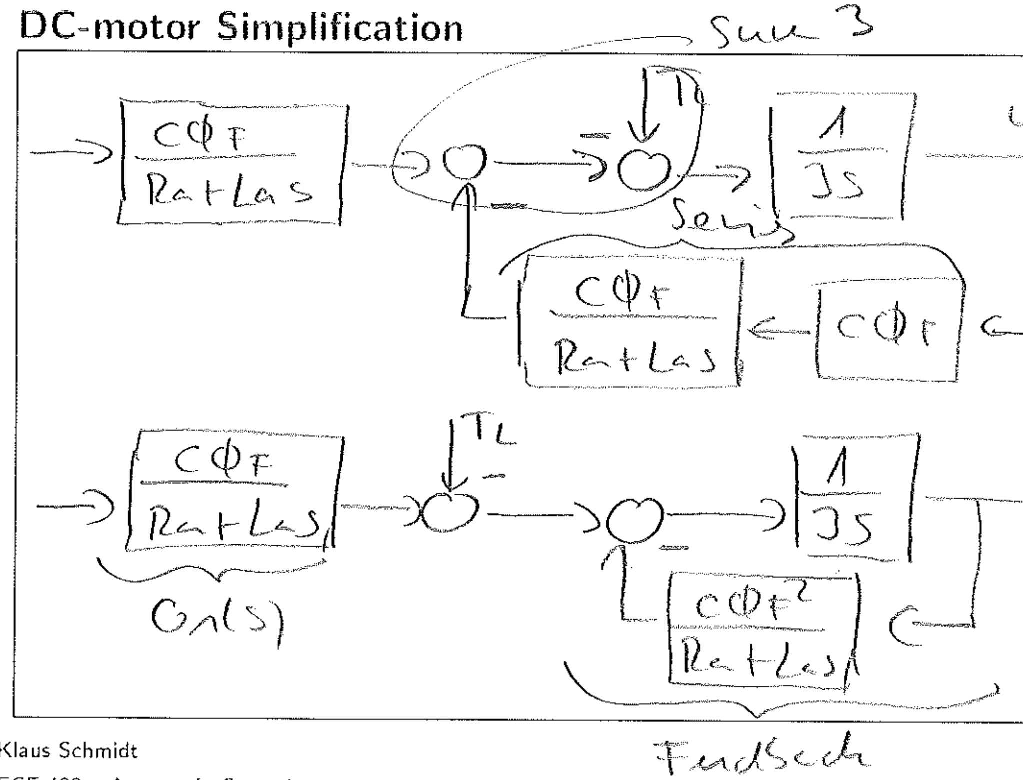
Block Diagram Simplification: Example

DC-motor Simplification



Block Diagram Simplification: Example

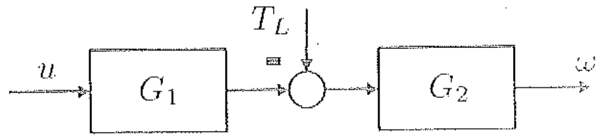
DC-motor Simplification



$$G_2(s) = \frac{\frac{1}{Js}}{1+\frac{1}{Js}\frac{C\Phi_F^2}{R_a+L_as}} = \frac{L_as+R_a}{JL_as^2+JR_as+C\Phi_F^2}$$

Block Diagram Simplification: Result

DC-Motor



Input-Output Behavior

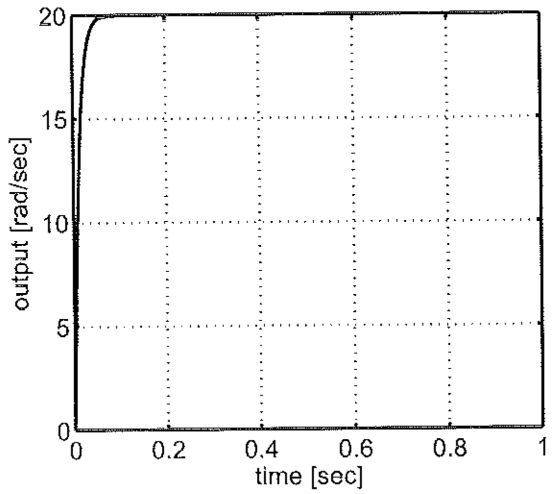
$$\frac{\Omega(s)}{U(s)} = G_1(s)G_2(s)$$

Disturbance-Output Behavior

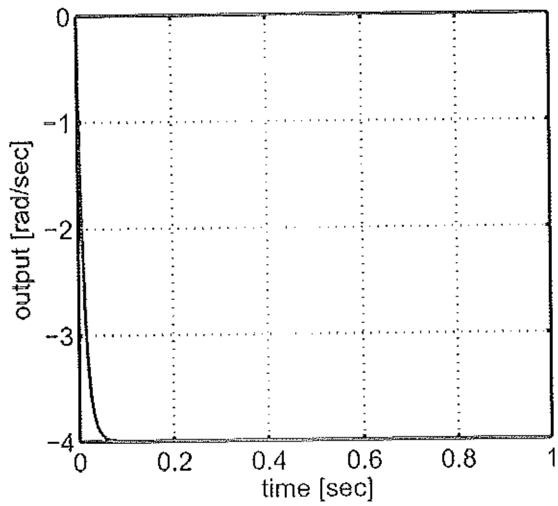
$$\frac{\Omega(s)}{T_L(s)} = G_2(s)$$

Block Diagram Simplification: Simulation

Input-Output Behavior



Disturbance-Output Behavior



Parameters

- $J_a = 3 \cdot 10^{-6}\,\mathrm{kg\,m^2}$; $R_a = 10\,\Omega$,
- $L_a = 2\,\mathrm{mH}$, $c\Phi_F = 0.05\,\mathrm{N\,m/A}$

Steps

- Input: voltage step of $1\,\mathrm{V}$
- Disturbance: torque $10^{-3}\,\mathrm{Nm}$