ECE 488 – Automatic Control

Controllability and State Feedback Control

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Compulsory Course in Electronic and Communication Engineering
Credits (3/0/3)

Course Webpage: http://ECE488.cankaya.edu.tr

Reminder

Previous Weeks

- Plant modeling
- Properties of transfer functions
- Stability and performance
- Feedback control
- Root locus method
- Nyquist criterion and bode plot
- Lead/lag compensator and PID-controller

This week

- Controllability
- State feedback control

Controllability: Preliminaries

State-space Model

$$\dot{x}(t) = A\,x(t) + b\,u(t)$$
$$y(t) = c^T\,x(t) + d_d\,u(t)$$

Transfer Function

$$G(s) = c_d^T(sI - A)^{-1}b + d$$

Solution of the State Equation

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}bu(\tau)d\tau$$

Example

Gap 1

$$\dot{x} = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = [0 \quad 1]\, x$$
$$G(s) = [0 \; 1] \begin{bmatrix} s & -3 \\ -1 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$
$$= [0 \quad 1] \frac{\begin{bmatrix} s+2 & 3 \\ 1 & s \end{bmatrix}}{s^2+2s-3} \begin{pmatrix} 1 \\ 0 \end{pmatrix} =$$
$$= \frac{1}{(s+3)(s-1)}$$

Controllability: Definition

Controllability Definition

A linear system is called completely controllable if it is possible for each pair of states x_1, x_2 to find a control input $u(t)$ that moves the system state from x_0 to x_1 in a specified transfer time t_T

Derivation of Controllability Condition

- From the solution of the state space equation

$$x_1 - e^{At_T}x_0 = \int_0^{t_T} e^{A(t_T-\tau)}bu(\tau)d\tau$$

Gap 2

Remember: $e^{At} = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \dots$

Thm Cayley-Hamilton:

$$e^{At} = \alpha_0(t)I + \alpha_1(t)A + \alpha_2(t)A^2 + \dots + \alpha_{n-1}(t)A^{n-1}$$

Controllability: Evaluation

Computation

Gap 3

[e^{At} can be written as linear combination of powers of A up to A^{n-1}]

$$\Rightarrow \int_0^{t_1} e^{A(t_1-\tau)} b\, u(\tau)\, d\tau =$$

$$= e^{At_1} \int_0^{t_1} \left(\alpha_0(-\tau) I + \alpha_1(-\tau) A + \dots + \alpha_{n-1}(-\tau) A^{n-1}\right) b\, u(\tau)\, d\tau$$

$$= e^{At_1}\Big(Ib \underbrace{\int_0^{t_1} \alpha_0(-\tau) u(\tau)\, d\tau}_{u_0} + Ab \underbrace{\int_0^{t_1} \alpha_1(-\tau) u(\tau)\, d\tau}_{u_1} + \dots + A^{n-1} b \underbrace{\int_0^{t_1} \alpha_{n-1}(-\tau) u(\tau)\, d\tau}_{u_{n-1}} \Big)$$

$$= e^{At_1} \underbrace{[b \;\; Ab \;\; \dots \;\; A^{n-1} b]}_{\mathcal{C}} \cdot \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix} = x_1 - e^{At_1} x_0$$

$\Rightarrow \mathcal{C}$ must be invertible to solve for u

Controllability: Verification

Controllability Test by Kalman

A linear system of order n is completely controllable if and only if the controllability matrix $C = \begin{bmatrix} b & A\,b & \cdots & A^{n-2}\,b & A^{n-1}\,b \end{bmatrix}$ has full rank n

$\mathcal{C}$

$\Rightarrow$ Check the rank of $\mathcal{C}$ to verify controllability

Example

Gap 4

$$A = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\rightarrow \mathcal{C} = [b \;\; Ab] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ has rank } 2$$

$\Rightarrow$ controllable

Stability: State Space Models

Definition

A linear system with the dynamic matrix A is asymptotically stable if all eigenvalues of A lie in the open left half plane

$\Rightarrow$ Stronger condition than BIBO stability: $G(s) = c^T(sI - A)^{-1}b + d$ can be BIBO stable even if A has eigenvalues in the right half plane!

Example

Gap 5

$A = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} \Rightarrow \det(\lambda I - A) = \det\begin{bmatrix} \lambda & -3 \\ -1 & \lambda+2 \end{bmatrix} =$

$= \lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1) = 0$

$\Rightarrow \lambda_1 = -3, \lambda_2 = +1$. One eigenvalue in the RHP $\Rightarrow$ unstable

State Feedback Control: Idea

Given

$$\dot{x}(t) = A\,x(t) + b\,u(t)$$
$$y(t) = c^T x(t) + d\,u(t)$$

Goal

- Use feedback of the state vector x to move the eigenvalues of the closed-loop system to desired locations

State Feedback

$$u(t) = k^T x(t) + M\,r(t)$$

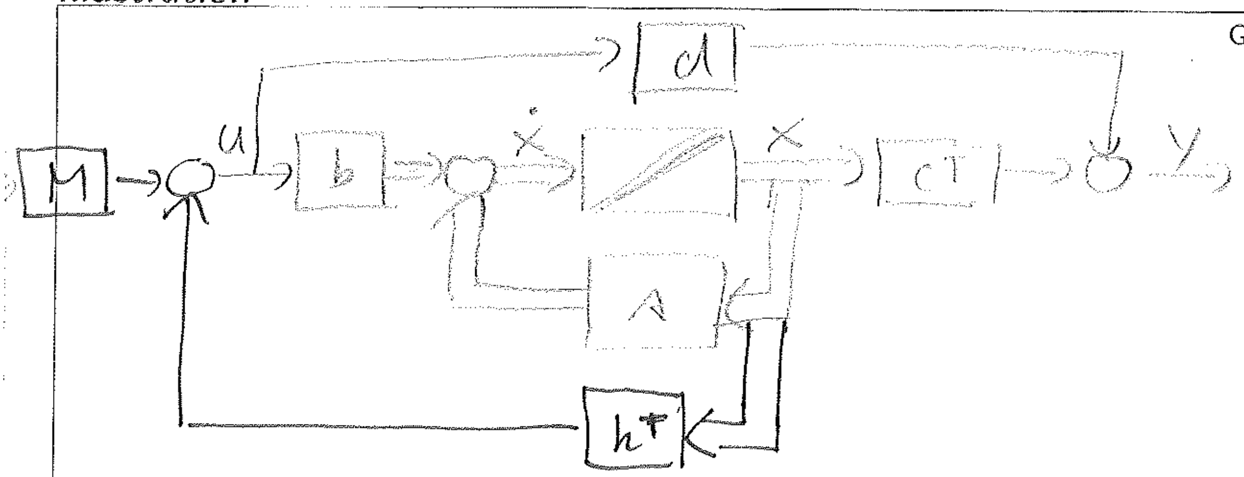
Design Parameters

- Feedback vector k; pre-filter M

State Feedback Control: Block Diagram

Illustration

Gap 6



Practical Fact

- All state variables must be measurable

State Feedback Control: Closed Loop

Computation

Gap 7

$$\dot{x}(t) = A\,x(t) + b\,(h^T x + M \cdot r) =$$
$$= (A + b h^T)\,x + b\,M \cdot r$$
$$y(t) = c^T x(t) + d\,(h^T x + M r) =$$
$$= (c^T + d\,h^T)\,x(t) + d\,M \cdot r$$

State Space Model

$$\dot{x}(t) = \underbrace{(A + b\,k^T)}_{\tilde{A}}\,x(t) + \underbrace{b\,M}_{\tilde{b}}\,r(t)$$

$$y(t) = \underbrace{(c^T + d\,k^T)}_{\tilde{c}^T}\,x(t) + \underbrace{d\,M}_{\tilde{d}}\,r(t)$$

Notation in Closed Loop

- Dynamic matrix
  $\tilde{A} = A + b\,k^T$
- Transfer function
  $\tilde{G}(s) = \tilde{c}^T(sI - \tilde{A})^{-1}\tilde{b} + \tilde{d}$

State Feedback Control: Closed Loop

Closed-loop Requirements

- Stability
  → All eigenvalues of $\tilde{A}$ should lie in the OLHP
- Sufficient performance
  → Suitable choice of the closed-loop poles eigenvalues (for example far away enough from the imaginary axis) using design parameter k
- Stationary exact tracking
  → Suitable choice of the design parameter M

Questions

- When is it possible to assign the poles of the closed loop?
- How can we compute the design parameters k and M?

State Feedback Control: Pole Assignment

Choice of the Pole Locations

If a linear system is completely controllable, then the eigenvalues of the closed-loop dynamic matrix $\tilde{A} = A + b\,k^T$ can be assigned arbitrarily by a suitable choice of k

Pole Assignment for Complete Controllability

- System order n: Choice of the closed loop characteristic polynomial (for example using desired pole locations)

$$p(s) = p_0 + p_1 s + \cdots + p_{n-1} s^{n-1} + s^n$$

→ We want that $p(s) = \det(sI - A_d - b_d k_d^T)$

State Feedback Control: Pole Assignment

Pole Assignment for Complete Controllability (alternative)

- Formula of Ackermann: Compute the vector v such that $v^T = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \mathcal{C}^{-1}$
- Compute state feedback vector k using $p(s)$ and v

$$k^T = -p_0 v^T - p_1 v^T A - \cdots - p_{n-1} v^T A^{n-1} - p_n v^T A^n$$

Example

Gap 8

$A = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}$; $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; closed-loop eigenvalues at $s = -4$ and $s = -2$

$\Rightarrow p(s) = (s+4)(s+2) = s^2 + 6s + 8 = p_2 s^2 + p_1 s + p_0$

controllability matrix:

$\mathcal{C} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow v^T = [0 \;\; 1]\, \mathcal{C}^{-1} = [0 \;\; 1] \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [0 \;\; 1]$

State Feedback Control: Example

Computation

Gap 9

State Feedback Computation

$k^T = -8\,[0 \;\; 1] - 6\,[0 \;\; 1] \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} - 1 \cdot [0 \;\; 1] \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}^2 =$

$= [0 \;\; -8] - [+6 \;\; -12] - [1 \;\; -2] \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} =$

$= [-6 \;\; 4] - [-2 \;\; 7] = [-4 \;\; -3]$

Eigenvalues of $A + bk^T$

$\det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} [-4 \;\; -3] \right) =$

$= \det \begin{bmatrix} \lambda+4 & 0 \\ -1 & \lambda+2 \end{bmatrix} = (\lambda+4)(\lambda+2)$ ✓

State Feedback Control: Uncontrollable Eigenvalues

Hautus Test

An eigenvalue λ of A is not controllable if and only if the matrix $\begin{bmatrix}(\lambda I - A) & b\end{bmatrix}$ does not have full rank n

$\rightarrow$ Eigenvalues λ of A that are not controllable cannot be moved

Pole Assignment by Comparison of Coefficients

- Determine the characteristic polynomial of the closed loop for $k^T = \begin{bmatrix}k_1 & k_2 & \cdots & k_n\end{bmatrix}$ ($k_1, \ldots, k_n$ are free parameters)
- We want

$$\det(sI - A - b\,k^T) = p_0 + p_1 s + \cdots p_{n-1}s^{n-1} + p_n s^n$$

- Compute the free parameters $k_1, k_2, \ldots, k_n$ by comparison of coefficients

State Feedback Control: Example

Computation

Gap 10

$A = \begin{bmatrix} 0 & 3 \\ 1 & -2 \end{bmatrix}$; $b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$, $\lambda_1 = -3$, $\lambda_2 = 1$

$\Rightarrow \begin{bmatrix}\lambda_1 I - A & b\end{bmatrix} = \begin{bmatrix} -3 & -3 & 3 \\ -1 & -1 & 1 \end{bmatrix}$ has rank 1

$\Rightarrow$ uncontrollable

$\begin{bmatrix}\lambda_2 I - A & b\end{bmatrix} = \begin{bmatrix} 1 & -3 & 3 \\ -1 & 3 & 1 \end{bmatrix}$ has rank 2

$\Rightarrow$ controllable

$\Rightarrow$ it is possible to change eigenvalue λ_2 but not λ_1

We want $\lambda_1 = -3$ and $\lambda_2 = -3$ in closed loop

State Feedback Control: Example

Computation

Gap 11

$p(s) = (s+3)^2 = s^2 + 6s + 9$

We write $k^T = [k_1 \ k_2]$ and compute

$\det(sI - A - bk^T) = \det\left(\begin{bmatrix} s & -3 \\ -1 & s+2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}[k_1 \ k_2]\right) =$

$= \det\begin{bmatrix} s-3k_1 & -3-3k_2 \\ -1-k_1 & s+2-k_2 \end{bmatrix} = (s-3k_1)(s+2-k_2) -$

$- (-1-k_1)(-3-3k_2) = (s+3)^2$

$\rightsquigarrow$ Choose for example $k_1 = -1$ and $k_2 = -1$

$\Rightarrow (s+3)(s+3) - 0 = (s+3)^2$

State Feedback Control: Pre-Filter

Goal

- For unit reference step, we want to reach stationary output value 1

Solution

- Apply final value theorem to the closed-loop transfer function:

$$
\begin{aligned}
1 &= \lim_{t\to\infty} y(t) = \lim_{s\to 0} \tilde{G}(s) \\
&= \lim_{s\to 0} \left((c^T + d\,k^T)(sI - A - b\,k^T)^{-1} b\,M + d\,M\right) \\
&= \left((c^T + d\,k^T)(-A - b\,k^T)^{-1} b + d\right) M \\
\Rightarrow M &= \frac{1}{(c^T + d\,k^T)(-A - b\,k^T)^{-1} b + d}
\end{aligned}
$$

$\rightarrow$ Note: eigenvalues of $-A - b\,k^T$ are non-zero

State Feedback Control: Example

Computation

Gap 12

Previous example with $c^T = [0\ 1]$

$$M = \frac{1}{[0\ 1]\left(\begin{bmatrix}0 & -3\\ -1 & 2\end{bmatrix} - \begin{bmatrix}3\\ 1\end{bmatrix}[-1\ -1]\right)^{-1}\begin{bmatrix}3\\ 1\end{bmatrix}} =$$

$$= \frac{1}{[0\ 1]\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}^{-1}\begin{bmatrix}3\\ 1\end{bmatrix}} =$$

$$= \frac{1}{[0\ 1]\underbrace{\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}}_{9}\begin{bmatrix}3\\ 1\end{bmatrix}} = \frac{9}{[0\ 3]\begin{bmatrix}3\\ 1\end{bmatrix}} = 3$$

State feedback: $u(t) = [-1\ -1]\,x(t) + 3\,r(t)$